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# Mathematical Reviews

*Edited by*

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# Mathematical Reviews

Vol. 14, No. 4 APRIL, 1953 Pages 341-438

## HISTORY

✶**Becker, Oskar, und Hofmann, Jos. E. Geschichte der Mathematik.** Athenäum-Verlag, Bonn, 1951. 340 pp. DM 10.00.

This book belongs to the section comprising the natural sciences in a comprehensive history of thought (Geschichte der Wissenschaften, II, Naturwissenschaften). It is packed with information and is a readable account of the development of mathematics throughout the centuries. For the mathematician it should be most useful and enlightening, however familiar he may be with certain periods, for he is likely to find much that is both novel and interesting in the book. After a general introduction it falls into three parts: (i) (pp. 13–113) upon ancient mathematics by Oskar Becker; and then both (ii) (pp. 117–141) upon Oriental mathematics and (iii) (pp. 142–264) upon Western mathematics by Joseph E. Hofmann. The first part covers Egyptian, Sumerian, Babylonian, Ancient Indian, Greek, Byzantine, and Roman; the two latter cover Indian, Moslem, Chinese, Japanese, and Western mathematics.

In their foreword (pp. 1–12) the authors describe the book as a sketch designed for those who already know something of the subject. They acknowledge that mathematics is more strongly spiritual than the natural sciences, that its history admits of a practical and a morphological treatment, being both matter of fact and also a part of the culture of a people. The authors have written with these two aspects in mind, regarding them not as contradictory but as complementary. Certain broad principles are laid down, as that Ancient India formulated general propositions but without proof; to Greece we owe the proof based upon definition, postulate and axiom, to early western mathematicians the formula, and to late western the logical calculus; all of which is preceded by a wealth of particular results found in Egypt, Babylonia, and the East. Interesting examples are given to illustrate these last: from Egypt came arithmetical approximations for the area of the circle (one being a square on 8/9 diameter); so, too, did similarity, symmetry, and repeated pattern, and the volume of a pyramid in terms of its height and base; from Babylon, tables of squares, cubes, and of a rudimentary Pascal arithmetical triangle, and a rudimentary mixture of simple and compound interest.

A remarkable passage deals with mathematics in Japan following the contact between East and West through the Jesuit mission towards the end of the 16th century: $\pi$ was calculated to 8 or 9 decimal places in 1674 by using a regular polygon of $2^{17}$ sides, and to 51 places in 1739 by Matsunaga, who had been preceded by Seki (1642–1708) who gave (one wonders in what notation) the first four terms of the series for $(\sin^{-1} x)^2$ in powers of $x$, based on an iterative approximation for a root of a certain quadratic equation. One would have liked to know something more about the notation of bygone days, granted that the ideas were the same as those implied by a modern idiom: thus it causes surprise to see 9th century arithmetic of India elucidated by the formula $(\sum a_k)^2=\sum a_k^2+2\sum a_p a_q$. But naturally in such a compact work obviously something had to be omitted.

The book appears to emphasize the arithmetical and analytical side of mathematics rather than the geometrical and kinematical, despite the importance of the latter as a vehicle for discoveries in the former, as for example in that of logarithms.

The treatment is masterly throughout: the story is continued in reasonable completeness to the latter part of the 19th century, with some hints about what has happened since. Hilbert, for example, is mentioned, but mainly for his influence on mathematical logic and not for his other pioneering achievements. Among the novelties that distinguish this book from former historical treatises is the promotion of James Gregory (1638–1675) to a place in the first rank of the founders of modern mathematics, with Descartes, Fermat, Wallis, Pascal, Huygens, Newton, and Leibniz.

The bibliography is excellent, a mine of useful information and references, as is also the index of names. A corresponding index of topics, which is lacking, would greatly add to the usefulness of the book. *H. W. Turnbull* (Millom).

✶**von Krbek, Franz. Eingefangenes Unendlich. Bekenntnis zur Geschichte der Mathematik.** Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1952. vii+331 pp.

The author compares his work to a series of excursions in a landscape with paved highways, but following older trails in order to experience the problems and difficulties of the pioneers. His book has thus become a series of essays which give an idea of the way in which our present great mathematical structures came into being. As a guide along the trails the author is lucid and entertaining, spicing his accounts with witticisms and anecdotes on the verge of prolixity. The different chapters give, in their rambling way, a good deal of solid information on the genesis of such subjects as projective and non-euclidean geometry, the rise of modern algebra, the beginnings of the calculus in the seventeenth century, differential geometry, the theory of point sets and topology. Contact with modern work is preserved, and several chapters, such as that on topology, may be useful to students looking for a readable orientation in this field. There are 125 figures and an instructive and critical index of source material; one footnote deals with the relations between the Riemann and Lebesgue integral in an interpretation of ideas of Baire and Hadamard.

*D. J. Struik* (Cambridge, Mass.).

✶**Dugas, René. Histoire de la mécanique.** Editions du Griffon, Neuchatel, 1950. 651 pp. 67.60 Swiss francs.

The view of the history and meaning of mechanics implied by most text books is derived from the quite unreliable "Science of mechanics" [5th ed., Open Court, Chicago,

1942] by E. Mach, which completely overlooks the deeper conceptual problems, dismissing them as mere "mathematics." There have been few subsequent attempts to treat the subject at large. Among them should be mentioned the relevant parts of the excellent and reliable but condensed article on the history of physics by E. Hoppe [Handbuch der Physik, Bd. 1, Springer, Berlin, 1926, pp. 1–179] and Duhem's original but rather overdrawn "Les origines de la statique" [2 vols., Hermann, Paris, 1905–06]. To correct the really remarkable inaccuracy of most of the historical statements in current books the best course is to return to the sources. Here E. Jouguet has done great service in making many short extracts available in his anthology, "Lectures de mécanique" [2 vols., Gauthier-Villars, Paris, 1922].

The author has taken up the monumental task of reading all the sources and forming a new picture of the entire development of mechanics from Aristotle's "Physics" to the quantum mechanics of the 1930's. His work is divided into five books. The first four, each of about 100 pages, consider: the "precursors," ending with Kepler; the formation of classical mechanics, from Stevin to Varignon; the organization and development of principles, from John Bernoulli to Lagrange; and selected characteristic later work, from Laplace to Duhem. The fifth book, 180 pages long, is about equally divided between relativity and quantum mechanics. The material is arranged in approximately chronological order.

To give a more detailed summary of what the reader will find in this work is not feasible. It will be necessary to restrict attention to the author's method and view of history, besides taking up a few particulars.

On pp. 623–625 we find: "Écrire l'histoire, de même qu'enseigner, c'est avant tout choisir. . . . Nous avons, à travers ce livre, multiplié les citations des textes originaux, ne les commentant que pour les éclaircir quand cela nous apparaissait, à tort ou à raison, nécessaire. . . . L'essentiel est que le lecteur . . . puisse être replacé dans le climat du siècle et dans le chemin, semé d'embûches, que les inventeurs *ont suivi*. Je souligne, car au XVIII$^e$ siècle, Clairaut parlait déjà, en ses ouvrages didactiques, du chemin que des inventeurs *eûssent pu suivre*. Cette école de la facilité n'a rien à voir avec l'histoire. Irai-je jusqu'à dire que je préfère les premiers classiques, si difficiles à lire parfois, pour le fait même de la peine qu'ils nous donnent à prendre contact avec une notion nouvelle? . . . Je me suis dispensé, également, de philosopher sur les principes de la mécanique en marge de l'histoire. C'est là un sujet d'étude qui présente un intérêt certain, mais il met en scène le critique, au lieu des acteurs mêmes de la science positive. La personnalité de l'historien risque d'être encombrante, son véritable rôle étant de sélection et non d'appréciation. Je ne me suis pas interdit, chemin faisant, quelques appréciations que l'on voudra bien me passer, *mais le plus souvent j'ai laissé le lecteur libre de juger sur pièces* [Reviewer's italics]. Les discussions que j'ai retracées sont dues, pour la plupart, à de vrais créateurs. Elles ont un caractère positif dans la mesure où elles ont annoncé, ou simplement permis, une extension des principes. Les époques où la science se borne à épuiser les conséquences de prémisses déterminées sont des périodes d'incompréhension latente. Faute d'avoir à remettre en question les bases mêmes de départ, on finit par s'endormir dans une sécurité trompeuse. . . . Je n'ai pas la prétention de convaincre ceux qui, par principe, estiment que l'histoire des sciences est un culte désuet et que chaque génération nouvelle doit, sans aucun régard en arrière, prendre le plus vite possible ses bases de depart dans la science positive. . . . Rien n'est futile en matière scientifique, pas même la contemplation du passé. Car celui-ci recèle la leçon de nos détours, de nos scrupules, de nos illusions et de nos erreurs. La science n'a jamais progressé de ce pas harmonieux dont il est aisé de donner l'illusion après coup. La connaissance directe des oeuvres anciennes . . . ne fait qu'enrichir les perspectives d'avenir qui s'ouvrent devant nous."

To this the reviewer can add only that the author has followed his program with scrupulous reserve and entire success. The original analysis is presented faithfully but in condensed form, with frequent quotations. The symbolism is often an intelligent compromise between the original and that now customary. Every line is relevant, instructive, and a pleasure to read. In numerous cases an evaluation of one master's work by another is quoted. The author's own comments usually explain a difficult passage, add historical background, or point out subsequent usage. His rare critical remarks are penetrating and sometimes urgently necessary, as when he says "la Renaissance affaiblira plutôt la mécanique du moyen âge, en provoquant un retour aux traditions classiques," and when he points out the metaphysical bias of Mach, the declared enemy of metaphysics.

This work is the first single volume from which it is possible to learn what rational mechanics really is. At a time when mechanics is reduced by one group of scientists to a mere subset of the theory of differential equations, by another to a few carelessly stated empirical rules and formal manipulations filling an introductory chapter in a text on quanta, this book presents positive evidence of the falsity of both these views. Needless to say, it describes the entirety of mechanical objects: not only mass-points, but bodies in the general sense, fluids and solids being included. Kinetic theory, statistical mechanics and thermodynamics are excluded. The intersection with Mach's book is nearly zero. Experiments are discussed only when they are really relevant, as in the case of Foucault's pendulum and gyroscope. But it is particularly instructive to learn from the actual historical data that in the development of relativity and quantum mechanics experiment played the same relatively minor, but of course necessary, part which it did in the history of classical mechanics; that all parts of mechanics have been wrought from experience by intellect, not measurement.

The reviewer hazards the conjecture that Mach's empiricism and its successor, operationalism, have run their destructive course. Further, that just as it is a characteristic feature of the modern view of the arts to present them in full consciousness of their history, so also in the mathematical sciences the historical view is to be the view of the next half century. The present book is a herald of this new trend. Destined no doubt to be ignored today, it may well be the handbook of future students of mechanics. (In a related field, reference should be made to the even more scholarly work of J. R. Partington [An advanced treatise on physical chemistry, 3 vols., Longmans Green, London-New York, 1949–1952]).

Lest the foregoing appear adulatory, the reviewer will now point out a few of the numerous omissions and resulting inaccuracies of the work. A better case could be made for Aristotle as a scientist, and even as a physicist [e.g., I. B. Cohen, Amer. J. Phys. 18, 343–359 (1950)]. In stating that the analysis in D'Alembert's "Essai . . . de la résistance des fluides," Paris, 1752, is "si longue et si difficultueuse que nous ne pouvons songer à la résumer ici," the author misses an important point. In the first place, D'Alembert's analysis

refers almost entirely to axially symmetric flows, not to plane ones, contrary to the statement made in all historical works and repeated by the author. Second, D'Alembert gives two derivations for the dynamical equation. In the first (§45), based essentially on the Bernoulli equation for stream-lines with the added assumption of uniform speed at $\infty$, he obtains the condition of potential flow. In the second (§48), based on "D'Alembert's principle," he obtains a more general result, from which he fallaciously concludes that the flow must be a potential flow. Thus D'Alembert himself gives an example of the unreliability of his "principle" in cases where the answer is not known in advance by other means. In discussing D'Alembert's paradox, the author does not tell us that Euler [Opera omnia, ser. 2, vol. 14, Teubner, Leipzig-Berlin, 1922, Ch. 2, Law I, remark 3] had discovered and proved it more directly seven years earlier. The author's attribution of the "Lagrangian description" to Lagrange rather than to Euler is inexcusable, since correct information in this regard is available even in the usual treatises. The author presents as due to Lagrange in 1788 the definitive proof that potential flow is only a special case, while in fact the very example Lagrange used is taken directly from a paper of Euler published in 1757. In presenting Lagrange's own proof of the velocity potential theorem, the author neglects to point out that it is false. Lagrange's only original contribution to the principles of hydrodynamics was reconciliation of Euler's equations with D'Alembert's principle in variational form, but this analysis (1761) the author does not mention at all. Although he devotes a whole chapter to the principle of least action, nowhere therein does he mention that the trajectories compared must have the same energy; while the analysis of Lagrange which the author reproduces (p. 331) employs this restriction, the author does not draw attention to it. There is no material whatever on the general properties of oscillating systems, and nothing on the mathematical theory of bodies moving in resisting media; in particular, there is no discussion of Book II of Newton's "Principia", the most original part of the whole work, though also largely incorrect. Three pages on rigid body mechanics seem hardly sufficient; and the notion of torque is barely mentioned. The author's selection of Cauchy's paper on finite local rotation of a continuous medium (1841) indicates finesse; but his omission of any account of Cauchy's discovery of the stress tensor and its properties is close to criminal.

In general, the author appears to have followed Jouguet's selection too closely. But here too the sin committed is of omission. The author's method of presenting the actual material of mechanics carries the corollary that the page as it stands is good without exception. *C. Truesdell.*

***Abetti, Giorgio. The history of astronomy.** Translated from the Italian by Betty Burr Abetti. Henry Schuman, New York, N. Y., 1952. xii+338 pp. $6.00.

This is the second history of astronomy published in English within a short span of time, after a lapse of about forty years. This book, by the director of the Astrophysical Observatory of Arcetri, is of a size comparable to that of Doig [A concise history of astronomy, Philosophical Library, New York, 1951; these Rev. 13, 809], addresses itself to a similar public, and is also good reading. Doig's book devotes proportionally more space than Abetti's to nineteenth and twentieth century development; Abetti's book pays more attention to the mathematical side, without going much into technical details. It also has illustrations, 41 in all. A chapter is devoted to the birth of astrophysics, another to the location, equipment, and methods of modern observatories. A special appendix adds some more material to this subject. The author stresses the necessity of international collaboration; the last chapter of his book is devoted to the International Astronomical Union.

*D. J. Struik* (Cambridge, Mass.).

**Uhler, Horace S. A brief history of the investigations on Mersenne numbers and the latest immense primes.** Scripta Math. 18, 122–131 (1952).

**Bliss, G. A. Autobiographical notes.** Amer. Math. Monthly 59, 595–606 (1952).

**Koyré, Alexandre. La mécanique céleste de J. A. Borelli.** Rev. Hist. Sci. Appl. 5, 101–138 (1952).

**Sauvenier-Goffin, Elisabeth. Note au sujet des manuscrits de H. Bosmans relatifs à Grégoire de Saint-Vincent.** Bull. Soc. Roy. Sci. Liège 21, 301–302 (1952).

***Cartan, Elie. Œuvres complètes. Partie I. Groupes de Lie.** Gauthier-Villars, Paris, 1952. Vol. 1: xxxii+pp. 1–568; vol. 2: viii+pp. 569–1356. Bound 5,500 francs; unbound 4,800 francs.

The following notice from the beginning of vol. 1 describes the organization and content of these collected works.

Cette édition des Œuvres d'Elie Cartan comprendra la totalité des notes et mémoires, mais non les volumes parus en librairie. Les travaux ont été groupés de la manière suivante: Partie I: Groupes de Lie. Partie II: Algèbre; systèmes différentiels et problèmes d'équivalence. Partie III: Géométrie différentielle, divers.

A l'intérieur de chaque Partie, les travaux seront classés par ordre chronologique. Chacune des trois Parties comportera plusieurs volumes. Au début de chaque volume, le lecteur trouvera la table des matières complète de la Partie dans laquelle est inclus ce volume; cette table donnera les références aux mémoires originaux.

Le présent volume contient en outre une liste chronologique complète des travaux. En face de chaque titre, le lecteur trouvera l'indication de la Partie dans laquelle sera inséré le travail correspondant.

L'édition a été faite par reproduction photographique des originaux ou du volume des "Selecta" [Gauthier-Villars, Paris, 1939]. Elle ne comporte ni corrections, ni commentaires.

**Sebastião e Silva, José. Obituary: Guido Castelnuovo.** Gaz. Mat., Lisboa 13, no. 52, 1–3 (1952). (Portuguese)

**Le Lionnais, F. Descartes et Einstein.** Rev. Hist. Sci. Appl. 5, 139–154 (1952).

**De Lorenzo, Giuseppe. Influsso di Galileo e di Kepler su Hobbes e Kant.** Rend. Accad. Sci. Fis. Mat. Napoli (4) 14 (1946–47), 182–186 (1948).

**Kucharski, W. Über Hamels Bedeutung für die Mechanik.** Z. Angew. Math. Mech. 32, 293–297 (1952).

***Dijksterhuis, E. J. Christiaan Huygens (Bij de voltooiing van zijn Œuvres Complètes). [Christiaan Huygens (On the completion of his Œuvres Complètes).]** De Erven F. Bohn N. V., Haarlem, 1951. 29 pp.

Lecture given in May, 1950.

**Agostini, Amedeo. Una lettera inedita di Goffredo Leibniz.** Arch. Internat. Hist. Sci. (N.S.) 5, nos. 18–19, 3–5 (1952).

**★Turnbull, Herbert Westren. Bi-centenary of the death of Colin Maclaurin (1698–1746), mathematician and philosopher, professor of mathematics in Marischal College, Aberdeen (1717–1725).** The University Press, Aberdeen, 1951. 20 pp.

**Pascal, Mario. Obituary: Roberto Marcolongo.** Rend. Accad. Sci. Fis. Mat. Napoli (4) 15 (1948), 1–12 (1 plate) (1949).

Marcolongo was born in 1862 and died in 1943. A list of his published works is included.

**★Markov, A. A. Izbrannye trudy. Teoriya čisel. Teoriya veroyatnosteĭ. [Selected works. Theory of numbers. Theory of probability.]** Izdat. Akad. Nauk SSSR, Leningrad, 1951. 719 pp. 32.00 rubles.

In addition to the "Selected works", edited by Yu. V. Linnik, there is a biography by Markov's son, an essay by Linnik, Sapagov and Timofeev on Markov's work in the theory of numbers and the theory of probability, commentary on the papers by these authors and Sarmanov, and a rather complete bibliography of Markov's published works and of papers concerning him.

**★Andrade, E. N. da C. Isaac Newton.** Chanticleer Press, New York, N. Y., 1950. 111 pp. (8 plates).

**★Poincaré, Henri. Science and method.** Translated by Francis Maitland. With a preface by Bertrand Russell. Dover Publications, Inc., New York, 1952. 288 pp. (1 plate). $1.25 paperbound, $2.50 clothbound.

Reprinted by photo-offset from the edition of Thomas Nelson and Sons, London, 1914. The preface by Bertrand Russell, indicated on the title page and included in the 1914 edition, has actually been deleted from this edition.

**★Poincaré, H. Science and hypothesis.** With a preface by J. Larmor. Dover Publications, Inc., New York, 1952. xxviii+244 pp. (1 plate). $1.25 paperbound, $2.50 clothbound.

Reprinted by photo-offset from the edition of the Walter Scott Publishing Co., London, 1905.

**Perron, Oskar. Obituary: Alfred Pringsheim.** Jber. Deutsch. Math. Verein. 56, Abt. 1, 1–6 (1952).

Essentially the same as the obituary in Jber. Bayer. Akad. Wiss. München 1944/48, 187–193 (1948); these Rev. 11, 573.

**Conte, Luigi. Vincenzo Riccati e il caso irriducibile dell'equazione cubica.** Period. Mat. (4) 30, 125–128 (1952).

**Straneo, Paolo. Obituary: Carlo Severini.** Boll. Un. Mat. Ital. (3) 7, 98–101 (1952).

**Agostini, Amedeo. I baricentri e loro proprietà in Leonardo da Vinci.** Boll. Un. Mat. Ital. (3) 7, 321–327 (1952).

**Giacomelli, Raffaele. La dinamica di Leonardo da Vinci.** Aerotecnica 22, 178–191 (1952).

**Aitken, A. C. Obituary: J. H. Maclagan Wedderburn, F. R. S.** Edinburgh Math. Notes 38, 19–22 (1952).

**Turnbull, H. W. Obituary: John Williamson.** Edinburgh Math. Notes 38, 23–24 (1952).

## FOUNDATIONS

**Ackermann, Wilhelm. Widerspruchsfreier Aufbau einer typenfreien Logik. (Erweitertes System).** Math. Z. 55, 364–384 (1952).

The author sets up a type-free system of logic which can be proved consistent. This system is an extension of the system described in an earlier paper [J. Symbolic Logic 15, 33–57 (1950); these Rev. 12, 384] but certain special cases of the law of the excluded middle are provable in the new system. An example is the statement '$a\equiv b \vee \overline{a\equiv b}$'. To carry out the consistency proof, it is convenient to use another system, equivalent to the given one. This new system is phrased in terms reminiscent of the work of Gentzen. The proof is similar to the one in the paper cited above, but due to the increased complexity of this system, changes are necessary. The development of mathematics on the basis of this system will follow in a future paper. *I. Novak Gál.*

**Surányi, János. Contributions to the reduction theory of the decision problem. V. Ackermann prefix with three universal quantifiers** Acta Math. Acad. Sci. Hungar. 2, 325–335 (1951). (Russian summary)

The author proves that given any formula of the first order predicate calculus, there exists an equivalent binary first order formula with prefix of the form $(Ex_1)(y_1)(Ex_2)(y_2)(y_3)$, and also one with prefix of the form $(y_1)(Ex_1)(Ex_2)(y_2)(y_3)$, each containing at most seven binary predicates. This result is similar to that of part II [Surányi, same Acta 1, 261–271 (1950); these Rev. 13, 715], which considers prefixes of the form $(x_1)(x_2)(x_3)(Ey)$ and $(x_1)(x_2)(Ey)(x_3)$, in that the total number of quantifiers and binary predicates is specified.

*O. Frink* (State College, Pa.).

**Kalicki, Jan. On Tarski's matrix method.** Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 41 (1948), 130–142 (1950). (English. Polish summary)

Suppose $m_1$ and $m_2$ are logical matrices as defined by Łukasiewicz and Tarski [C. R. Soc. Sci. Varsovie 23, 30–50 (1930)]. Let $E(m_1)$, $E(m_2)$ be the sets of well-formed expressions which satisfy $m_1$, $m_2$, respectively. The author then defines an equivalence relation $m_1\equiv m_2$, the sum $m_1+m_2$, the product $m_1\cdot m_2$, and also a partial ordering of matrices. Addition and multiplication are associative and commutative, and there is a unit for each. Also $E(m_1+m_2)=E(m_1)+E(m_2)$, $E(m_1\cdot m_2)=E(m_1)\cdot E(m_2)$. *I. Novak Gál* (Ithaca, N. Y.).

**Kalicki, Jan. A test for the equality of truth-tables.** J. Symbolic Logic 17, 161–163 (1952).

In previous papers [same J. 15, 174–181, 182–184 (1950); these Rev. 12, 663] the author has defined the notion of equivalence (which he calls equality) of two truth-tables. In this paper he derives a necessary and sufficient condition for equality. Introducing a new type of truth-table with "super-designated" elements as well as designated and undesignated elements, he defines a kind of product of two truth-tables

$A$ and $B$, and shows that $A$ and $B$ are equal if and only if every well-formed formula is a tautology according to this product. This leads to a decision procedure, and gives a theoretical solution of most of the questions raised in the previous papers. *O. Frink* (State College, Pa.).

**Greniewski, H. Arithmetics of natural numbers as part of the bi-valued propositional calculus.** Colloquium Math. **2** (1951), 291–297 (1952).

Consider the notations for natural numbers in the binary system; we shall call such a notation a binary number. Since any binary number is a sequence of 0's and 1's, and since the binary propositional algebra is an algebra of 0 and 1, it is natural to expect that an arithmetic of binary numbers can be set up by making appropriate definitions on the basis of this propositional algebra. The author here shows in detail how this can be done. For example, he defines for each $m$ and $n$ (by a double induction) a function $f(p_1, \cdots, p_m, q_1, \cdots, q_m)$ which is 1 or 0 according as the sequences $p_1 \cdots p_m$ and $q_1 \cdots q_n$ represent the same or distinct binary numbers; likewise the sum of the binary digits $p$ and $q$ is the pair $rs$ where $r$ is the conjunction of $p$ and $q$ and $s$ is the symmetric difference, and from this the sum of the binary numbers can be defined by induction. Definitions of products and powers are included. The utilization of his definitions for digital computing machinery is to be discussed elsewhere. *H. B. Curry* (State College, Pa.).

**Łoś, J. An algebraic proof of completeness for the two-valued propositional calculus.** Colloquium Math. **2** (1951), 236–240 (1952).

The completeness of the two-valued propositional calculus (i.e., the derivability from accepted axioms of every tautology) has been proved by various persons. The proof given here differs from the others in its "algebraic" character. The central idea is that if $a$ is a formula which is not a tautology, than there is a homomorphism $h(x)$ of the system on the two-valued truth table such that $h(a)=0$. The establishment of this homomorphism requires rather complex reasoning; in fact, on the basis of an enumeration of all formulas, the author establishes a sequence of systems $A_1, A_2, A_3, \cdots$, and then takes the union of all of them. (The author is in error in stating that the first proof of completeness appeared in a publication of Łukasiewicz [Elementy logiki matematycznej, Warsaw, 1929]; the first published proof, due to Post, appeared in 1921 [Amer. J. Math. **43**, 163–185 (1921)].) *H. B. Curry* (State College, Pa.).

**Singh, R. K. P., and Shukla, R. A note on Götlind's axiom system for the calculus of propositions.** Math. Student **18** (1950), 108–110 (1951).

The authors assert that the following axiom system, due to E. Götlind [Norsk Mat. Tidsskr. **29**, 1–4 (1947); these Rev. **9**, 1], is independent: (a) $p \vee p \to p$; (b) $p \to p \vee p$; (c) $p \to p$; (d) $p \to r. \to: q \vee p. \to. r \vee q$. Actually (c) follows from the other axioms, as has been shown by H. Rasiowa [ibid. **31**, 1–3 (1949); these Rev. **11**, 303]. The flaw in the independence proof is pointed out in the note reviewed below. *B. Jónsson* (Providence, R. I.).

**Götlind, Erik. A note on an article by R. K. P. Singh and R. Shukla.** Math. Student **19** (1951), 120–121 (1952).

See the preceding review. *B. Jónsson.*

**Stenius, Erik. Das Interpretationsproblem der formalisierten Zahlentheorie und ihre formale Widerspruchsfreiheit.** Acta Acad. Aboensis **18**, no. 3, 102 pp. (1952).

In the present paper the author uses what might be called a semi-constructivist approach to the foundations of arithmetic in an attempt to ensure their consistency. Thus, existence theorems in the meta-language are conceived in a constructivist sense. On the other hand, in the object language—in the first instance the language ($Z$) of Hilbert-Bernays—existential propositions are defined in a familiar manner as negations of universal propositions. By the "interpretation" of the language the author means a syntactical definition of truth or falsehood for the propositions of the language which is to be in consonance with the deductive calculus within the language. More particularly, the propositions of $Z$ are divided into classes of rising integral order according to the maximum number of successive quantifications involved in them. The truth or falsehood of a quantified statement of positive order is then made to depend on the deductive properties of a set of corresponding statements of lower order. Gödel's first theorem of undecidability (non-deducibility) is avoided by means of this definition. On the other hand, not all well-formed propositions of ($Z$) can be interpreted in this way some of which remain therefore "undefined" (to use the author's terminology). Extensions to calculi of higher order are sketched but the limitations of the present method may become more pronounced for these cases. The author intends to deal with this problem in more detail in a future paper. *A. Robinson* (Toronto, Ont.).

**Rieger, Ladislav. On the Marxist conception of mathematics.** Časopis Pěst. Mat. **76**, 73–103 (1951). (Czech)

## ALGEBRA

**Maak, Wilhelm. Ein Problem der Kombinatorik in seiner Formulierung von H. Weyl.** Math.-Phys. Semesterber. **2**, 251–256 (1952).

Le problème de combinatoire en question est le suivant: A quelles conditions deux relations d'équivalence sur un même ensemble possèdent-elles un système de représentants communs? L'auteur montre d'abord que ce problème présente un intérêt particulier car certaines branches des mathématiques (en particulier: théorie des fonctions presque-périodiques et théorie ergodique) peuvent être établies d'une manière simple en faisant jouer à ce problème un rôle important et que sa démonstration fournit un exemple remarquable pour la méthodique de la pensée mathématique. Il fait ensuite une brève revue des différents progrès du problème depuis Miller (1910) (relations d'équivalence compatibles avec une structure de groupe fini), van der Waerden (1926) (relations d'équivalences sur un ensemble fini dont les classes ont même nombre d'éléments), Sperner (1927), P. Hall (1935) (condition nécessaire et suffisante), W. Maak (1936), D. König (1938) jusqu'à sa formulation en 1949 par H. Weyl sous forme de "problème de marriage" [cf. Halmos et Vaughan, Amer. J. Math. **72**, 214–215 (1950); ces Rev. **11**, 423]. Une revue plus complète aurait sans doute mentionné les noms de Sainte Lagüe (1921), Menger, R. Rado (1933), Gallai (Grünwald) (1938), Schmushkovitch (1939), Tutte (1947). (Pour un exposé

complet et les références voir l'ouvrage du rapporteur: Fondements de la théorie des relations binaires, Paris, 1951.)
*J. Riguet* (Paris).

**Carlitz, L. Congruences for the ménage polynomials.** Duke Math. J. **19**, 549–552 (1952).

The reviewer's congruence [same J. **19**, 27–30 (1952); these Rev. **13**, 616] for ménage polynomials, $U_n(t)$, which are generating functions for numbers $u_{n,r}$ associated with Lucas' problème des ménages, is shown to hold for a general modulus $m$. Results are also given for related polynomials, $V_n(t)$ which apply to the same problem with a straight rather than a circular table, and $W_n(t)$ which seem to have no combinatorial interpretation. For $m$ a prime power, the congruences for the polynomials entail periodic properties for the numbers, which are given explicitly. *J. Riordan.*

**Ryser, H. J. Matrices with integer elements in combinatorial investigations.** Amer. J. Math. **74**, 769–773 (1952).

L'auteur poursuit l'étude des "balanced incomplete block designs" symétriques, c'est-à-dire, pour lesquels le nombre des blocs est égal à celui des traitements ($v=b$) [Bruck et Ryser, Canadian J. Math. **1**, 88–93 (1949); ces Rev. **10**, 319; Ryser, Proc. Amer. Math. Soc. **1**, 422–424 (1950); ces Rev. **12**, 153; Chowla et Ryser, Canadian J. Math. **2**, 93–99 (1950); ces Rev. **11**, 306]. Soient $1, 2, 3, \cdots, n$ les $n$ variétés et $S_1, S_2, \cdots, S_n$ les $n$ blocs de $k$ variétés chacun, tels que deux quelconques d'entre eux aient toujours exactement $\lambda$ variétés en commun. Soit $A$ la matrice carrée $n\cdot n$ obtenue en écrivant un 1 dans la ligne $i$ et la colonne $j$ si la variété $j$ appartient au bloc $S_i$ et un zéro dans le cas contraire. Soit $B$ la matrice carrée d'ordre $n$ telle que tous les éléments de la diagonale principale soient égaux à $k$ et que tous les autres soient égaux à $\lambda$, et $A'$ la transposée de $A$. Alors $AA'=A'A=B$. Les réciproques suivantes sont établies: I) Si $B=AA'=A'A$, où $A$ est composée d'éléments entiers, alors $A$ fournit une solution du problème $n, k, \lambda$. II) Si $k$ est premier avec $\lambda$, $k-\lambda$ impair, $A$ formée d'éléments entiers et si $AA'=B$, alors $A$ fournit une solution du problème $n, k, \lambda$. Application aux cas: $k=N+1$, $n=Nk+1$ et $n=4N-1$, $k=2N-1$. *A. Sade* (Marseille).

**Schönhardt, Erich. Über positiv definite Matrizen.** Z. Angew. Math. Mech. **32**, 157–158 (1952).

In this note the author discusses three definitions of "positive definite" as used in current literature for non-symmetric real square matrices. He shows that these definitions are equivalent only in the case of symmetric matrices, and discusses their relationships when applied to non-symmetric matrices.
*H. Polachek* (Silver Spring, Md.).

**Sherman, S. On a conjecture concerning doubly stochastic matrices.** Proc. Amer. Math. Soc. **3**, 511–513 (1952).

A real $n\times n$-matrix $P=(p_{ij})$, $i, j=1, \cdots, n$, is doubly stochastic if $p_{ij}\geqq 0$ and $\sum_i p_{ij}=\sum_j p_{ij}=1$ for $i, j=1, \cdots, n$. Define a partial ordering of the set $\mathfrak{P}$ of all doubly stochastic $n\times n$-matrices by $P^1 \dashv P^2$ if there exists a $P^3 \in \mathfrak{P}$ such that $P^1=P^2P^3$. Also, define a partial ordering of the set $\mathfrak{A}$ of all real $n$-vectors $a=(a_1, \cdots, a_n)$ by $a \dashv b$ if $\sum_i\phi(a_i)\leqq\sum_i\phi(b_i)$ for any real convex function $\phi$. The author proves the following result conjectured by the reviewer: $P^1 \dashv P^2$ if and only if $P^1a \dashv P^2a$ for all $a \in \mathfrak{A}$. *S. Kakutani.*

**Bencivenga, Ulderico. Su di un tipo di algebra di ordine *n*.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **14** (1946–47), 11–17 (1948).

**Bencivenga, Ulderico. Su di un'espressione degli elementi di un tipo di algebra di ordine *n* mediante certe funzioni esponenziali.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **14** (1946–47), 29–35 (1948).

**Bencivenga, Ulderico. Su di un altro tipo di algebra di ordine *n*.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **14** (1946–47), 150–154 (1948).

The first two papers discuss in an elementary way the algebra of order $n$ generated by an element $u$ satisfying $u^n=1$, without observing that it is a direct sum of $n$ copies of the complex numbers. The third paper discusses a real algebra with basis elements behaving like $\cos(2k\pi/n)+\sin(2k\pi/n)$, $k=1, \cdots, n$. *I. Kaplansky* (Chicago, Ill.).

**Spampinato, Nicolò. Forma esplicita delle funzioni sen $\xi$, cos $\xi$, $e^\xi$ nel campo ipercomplesso.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **18** (1951), 219–226 (1952).

Explicit computation of the three functions in the title in the case of the "$n$-dual" algebra (a unit element adjoined to a trivial algebra), and the tripotential and quadripotential algebras (a unit element and a nilpotent element of index 3 or 4). *I. Kaplansky* (Chicago, Ill.).

**Rocco Boselli, Anna. Algebre complesse d'ordine 5 dotate di modulo.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **14** (1946–47), 170–182 (1948).

The author tabulates (with references to Scorza) five-dimensional algebras with unit element over the complex numbers. *I. Kaplansky* (Chicago, Ill.).

## *Abstract Algebra*

**Petresco, Julian. Théorie relative des chaînes. II. Isocorrespondance.** C. R. Acad. Sci. Paris **235**, 1087–1089 (1952).

The author considers the relation of his earlier paper [same C. R. **235**, 226–228 (1952); these Rev. **13**, 901] to the notions of transposes and projectivity in lattices. Intervals $A\leqq X\leqq A^*$ and $B\leqq Y\leqq B^*$ are called $\cap$-isocorrespondent if

$$A\cup\{A^*\cap[B\cup(B^*\cap X)]\}=X,$$
$$B\cup\{B^*\cap[A\cup(A^*\cap Y)]\}=Y.$$

Stronger forms of such correspondence are also defined. If both $\cap$-isocorrespondence and its dual hold, "$\cap$-" is omitted. *P. M. Whitman* (Silver Spring, Md.).

**Trevisan, Giorgio. Su una questione relativa alle strutture distributive.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **18** (1951), 144–145 (1952).

The author simplifies postulates for a ternary operation equivalent to distributivity in a lattice. [For references to more extensive simplifications, see the review of J. Hashimoto, Math. Japonicae **2**, 49–52 (1951); these Rev. **13**, 617.]
*P. M. Whitman* (Silver Spring, Md.).

**Aumann, Georg. Alternativ-Zerlegungen in Booleschen Verbänden.** Math. Z. **55**, 109–113 (1951).

The author reproves the following theorem due to M. H. Stone [Trans. Amer. Math. Soc. **40**, 37–111 (1936), p. 105]: In a Boolean ring $A$ every proper ideal is the product of all its prime ideal divisors. *B. Jónsson* (Providence, R. I.).

**Luce, R. Duncan. A note on Boolean matrix theory.** Proc. Amer. Math. Soc. **3**, 382–388 (1952).

Let $A$ be a Boolean (B) matrix of order $n$ and let $A'$ and $A^T$ stand for the complement and the transpose of $A$ and $0$, $I$, and $E$ for the zero, identity, and universal matrix. The author defines special types of B matrices and studies their properties. $A$ is said to be (i) symmetric, (ii) skew-symmetric, (iii) row (column) consistent, (iv) orthogonal if (i) $A^T \cap A' = 0$, (ii) $A^T \cap A = 0$, (iii) $AE = E$ $(EA = E)$, (iv) $A$ has an inverse which is $A^T$. Results: $A$ can always be uniquely decomposed into the disjoint union of a symmetric and a skew-symmetric matrix. $A$ has an inverse if and only if it is orthogonal; $A$ is involutory $(A^2 = I)$ if and only if it is both symmetric and orthogonal. In the last section the author considers the problem of finding the class of B matrices $X$ which are solutions of $XA = B$ $(AX = B)$, $A$ and $B$ being given B matrices. The problem of finding the solutions $X$ of $XA \subset B$ is completely solved; for the analogous problem $XA \supset B$ the author gives only a sufficiency condition for $X$ to be a solution of it. The author also presents a necessary and sufficient condition on $A$ and $B$ that there be solutions to $XA = B$: $B \subset (B'A^T)'A$. *O. Borůvka.*

**Rieger, Ladislav. On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem.** Czechoslovak Math. J. 1(76), 29–40 (1951) = Čehoslovack. Mat. Ž. 1(76), 33–49 (1951).

The author considers certain families $\Phi$ of multiple sequences over a Boolean algebra $B$. One of the conditions on $\Phi$ is that each multiple sequence has a sum and a product in $B$. It is shown that if $\Phi$ is countable, then $(B, \Phi)$ is isomorphic to a system $(F, \Psi)$ where $F$ is a set-field and $\Psi$ is a subfamily of $F$. Applied to the so-called Lindenbaum algebra of the lower predicate calculus, this yields a proof of Gödel's completeness theorem. *B. Jónsson.*

**Rieger, Ladislav. On free $\aleph_\xi$-complete Boolean algebras. (With an application to logic.)** Fund. Math. **38**, 35–52 (1951).

The author constructs an free $\aleph_\xi$ complete Boolean algebra $A_{\mathfrak{m}}^{\aleph_\xi}$ with $\mathfrak{m}$ generators. $A_{\mathfrak{m}}^{\aleph_\xi}$ is shown to be isomorphic to a countably complete set-field, in fact to the minimal σ-field of Borel subsets of the generalized Cantor's discontinuum. Analogous representations of $A_{\mathfrak{m}}^{\aleph_\xi}$ are impossible if $\aleph_\xi \geqq 2^{\aleph_0}$. These results imply Loomis' result that every countably complete Boolean algebra is isomorphic to a quotient $F/I$ where $F$ is a countably complete set-field and $I$ is a countably complete ideal. They also yield a solution to problems 79 and 80 of G. Birkhoff's Lattice theory [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; these Rev. 10, 673]. *B. Jónsson* (Providence, R. I.).

**Sikorski, Roman. A note to Rieger's paper "On free $\aleph_\xi$-complete Boolean algebras".** Fund. Math. **38**, 53–54 (1951).

Simple proofs of two of the results of the paper reviewed above. *B. Jónsson* (Providence, R. I.).

**Amitsur, S. A. A general theory of radicals. I. Radicals in complete lattices.** Amer. J. Math. **74**, 774–786 (1952).

This paper contains an axiomatic approach to the study of radicals in complete lattices. Since the lattice of all ideals in a ring is complete, generalizations are thus obtained of certain known properties of various radicals of a ring. If $M$ is a complete lattice, the unit and the zero of $M$ may be denoted by $I_M$ and $O_M$, respectively. A binary relation $\rho$ defined in $M$ is called an $H$-relation in $M$ if it satisfies the three following conditions: (A) If $a\rho b$; $a, b \in M$, then $a \geqq b$; (B) $a\rho a$ for $a \in M$; (C) If $a\rho b$ and $c \geqq b$, then $(a \cup c)\rho c$. Hereafter, all binary relations are assumed to be $H$-relations, and all sublattices are assumed to be complete. If $L$ is a sublattice of $M$ and $a$, $b$ are elements of $L$ such that $a\rho b$, we may say that $a$ is a $\rho$-element over $b$ in $L$. If $a \in L$ such that $a\rho O_L$, then $a$ is called a $\rho$-element in $L$. The sublattice $L$ is said to be $\rho$-semi-simple in $M$ if $L$ does not possess nonzero $\rho$-elements. If the $L$-interval $[a, b]$ is $\rho$-semi-simple in $M$, we write $a\bar{\rho}b$. If $a\bar{\rho}I_L$, $a$ is said to be a $\bar{\rho}$-element in $L$. An element $r \in L$ is called a $\rho$-radical in $L$ if $r$ is both a $\rho$-element in $L$ and a $\bar{\rho}$-element in $L$. By analogy with the definition of the lower radical of Baer [same J. 65, 537–568 (1943); these Rev. 5, 88], the author defines an upper $\rho$-radical $u(L, \rho)$ of $L$ in $M$. Every sublattice $L$ possesses at most one $\rho$-radical, denoted by $r(L, \rho)$. The upper $\rho$-radical of $L$ is the minimal $\bar{\rho}$-element of $L$. A necessary and sufficient condition that $r(L, \rho)$ exist is that $u(L, \rho)$ be a $\rho$-element, and then $r(L, \rho) = u(L, \rho)$. The $\rho$-radical exists in every lattice $L$ of $M$ if and only if for every $a, b \in L$ such that $a < b$ and such that $b$ is not a $\rho$-element over $a$, there is an element $c$ of $L$ such that $c\bar{\rho}b$ in $L$. Many other related results are obtained which cannot be mentioned in this review. The duality of the theory is brought out, and some examples are given showing connections with ring theory. As one example, let $M$ be the lattice of ideals of a ring $R$ and define $a\nu b$ if $a^n \leqq b \leqq a$ for some positive integer $n$. Then $\nu$ is an $H$-relation, the $\bar{\nu}$-elements are radical ideals as defined by Baer, and the $\nu$-radical (if it exists) is the nilpotent radical of the ring. *N. H. McCoy* (Northampton, Mass.).

**Marczewski, E. Sur les congruences et les propriétés positives d'algèbres abstraites.** Colloquium Math. **2** (1951), 220–228 (1952).

Développement de résultats déjà signalés par Garrett Birkhoff [Proc. Cambridge Philos. Soc. 31, 433–454 (1935)]: les "propriétés positives" d'algèbres se conservent par passage au quotient. L'auteur ne signale pas les travaux de Tarski et Krasner [cf. Krasner, Math. Pures Appl. (9) 17, 367–385 (1938); voir aussi J. Schmidt, Math. Nachr. 7, 165–182 (1952); ces Rev. 13, 904] qui permettraient de se placer à un point de vue beaucoup plus général. *J. Riguet.*

**Peremans, W. Some theorems on free algebras and on direct products of algebras.** Simon Stevin **29**, 51–59 (1952).

A necessary condition is given, and also a sufficient condition, for the existence of free algebras with a given set of operations and a given consistent system of axioms. *B. Jónsson* (Providence, R. I.).

**Evans, Trevor. Embedding theorems for multiplicative systems and projective geometries.** Proc. Amer. Math. Soc. **3**, 614–620 (1952).

The following theorems are proved. 1. Any countable loop can be embedded in a loop generated by one element. 2. Any countable semigroup can be embedded in a semigroup generated by two elements. 3. Any countable projective plane can be embedded in a projective plane generated by four points. 4. If every countable algebra of a class $\mathfrak{A}$ has the property $(E_1)$ that it can be embedded in an $\mathfrak{A}$-algebra generated by $m$ elements, then the free $\mathfrak{A}$-algebra $H$ on a countable numbers of generators can be embedded in a free $\mathfrak{A}$-algebra on $n \leqq m$ generators. Without altering the proof

of the last theorem, one may replace its hypothesis by the weaker condition that $E_1$ holds for $H$. *F. W. Levi.*

**✶Azumaya, Gorô. Tanjun-kan no daisûteki riron.** [Algebraic theory of simple rings.] Kawade-shobô, Tokyo, 1951. 2+2+182 pp. 300 Yen.

The theory of simple rings with minimum condition is presented, particularly making use of the author's method given in a former paper [Proc. Japan Acad. **22**, no. 11, 325–332 (1946); these Rev. **13**, 101]; the method is closely related to the one of Artin and Whaples [Amer. J. Math. **65**, 87–107 (1943); these Rev. **4**, 129] but is in certain respects more generally applicable. One of the features of the book is the author's outer Galois theory of simple rings [J. Math. Soc. Japan **1**, 130–146 (1949); these Rev. **11**, 414] (which is a generalization of Jacobson's theory (for sfields) [Ann. of Math. (2) **41**, 1–7 (1940); these Rev. **1**, 198] and which has been further extended by the reviewer [J. Math. Soc. Japan **1**, 203–216 (1949); these Rev. **12**, 237]). *T. Nakayama.*

**Clarke, F. Marion. Note on quasi-regularity and the Perlis-Jacobson radical.** Portugaliae Math. **11**, 89–94 (1952).

In order that the theory of the radical defined in terms of quasi-regularity should extend satisfactorily to non-associative rings some weak substitute for the associative law is required. In the development of the theory by Baer [Amer. J. Math. **65**, 537–568 (1943); these Rev. **5**, 88] and Jacobson [ibid. **67**, 300–320 (1945); these Rev. **7**, 2], at essential points where $xy\cdot z = x\cdot yz$ is used it would be enough to postulate this merely when $x$ is a left and $z$ a right quasi-inverse of $y$. This postulate (P) can be weakened further, and other ways of stating it are discussed. It is said that the alternative law expresses a stronger assumption of associativity than P; but it should be remarked that the alternative law as here interpreted $(x\cdot x^{-1}y = yx^{-1}\cdot x = y)$ includes the assumption of unique inverses. There is an interesting example which however does not have the asserted properties "every non-zero element is regular" and P. *I. M. H. Etherington* (Edinburgh).

**Waddell, Mathews C. Properties of regular rings.** Duke Math. J. **19**, 623–627 (1952).

A ring $R$ is regular if for each $a \in R$, there exists an $x \in R$ such that $axa = a$; it is biregular if each principal (two-sided) ideal is generated by an idempotent in the center of $R$; it is strongly regular if for each $a \in R$ there exists an $x \in R$ such that $a^2x = a$. Regular rings were introduced by von Neumann [Proc. Nat. Acad. Sci. U. S. A. **22**, 707–713 (1936)], biregular and strongly regular rings by Arens and Kaplansky [Trans. Amer. Math. Soc. **63**, 457–481 (1948); these Rev. **10**, 7]. The author shows that the ideals of a regular or biregular ring form a distributive lattice. If $R$ is a biregular ring in which every maximal ideal has a nonzero annihilator, then (i) the ideals of $R$ form a Boolean algebra, (ii) each nonzero ideal of $R$ contains a minimal ideal, and (iii) $R$ is the discrete direct sum of its minimal ideals. For a regular ring $R$, the author gives a number of conditions (several of them already known) which are equivalent to the condition that $R$ be strongly regular. *N. H. McCoy.*

**Kaplansky, Irving. Locally compact rings. III.** Amer. J. Math. **74**, 929–935 (1952).

In continuation of earlier investigations [same J. **69**, 153–183 (1947); **70**, 447–459 (1948); **73**, 20–24 (1951); these Rev. **8**, 434; **9**, 562; **12**, 584], the author presents three theorems. The first concerns non-discrete locally compact primitive rings of characteristic 0. It is shown that such a ring is a finite-dimensional algebra over its center. This is not true for characteristic $p$ [see the second reference above]; it is however true if the ring is also assumed simple and with minimal ideals. A ring is called simple here if it has no proper (not necessarily closed) two-sided ideals. The third result extends the previously obtained structure theorem of locally compact bounded semi-simple rings to only right bounded ones. While such a ring when bounded is the direct sum of a compact and a discrete ring, a right bounded ring contains an open ideal $J$ which is a local direct sum of finite simple rings relative to non-zero right ideals. The ring $A/J$ can be an arbitrary discrete semi-simple ring. A counter-example shows that $J$ need not be a direct summand. *O. Taussky-Todd* (Los Angeles, Calif.).

**Vandiver, H. S. A development of associative algebra and an algebraic theory of numbers. I.** Math. Mag. **25**, 233–250 (1952).

This is the first of a number of papers in which the author plans to treat the foundations of arithmetic and algebra and carry this treatment through elementary number theory to a discussion of algebraic numbers. He begins by giving a postulational development of a kind of generalized associative algebra of what he calls "combinations". A combination is essentially a linearly ordered set of symbols involving, among others, symbols of conjunction ("+" and "×") and parenthesis symbols: "(" and ")". The relation of equality "=" is also undefined and the essential postulate is a certain rule of substitution. Closure under addition and multiplication is proved rather than defined. In the course of the discussion the natural numbers are introduced purely symbolically and examples of unusual algebras of this type are also given. *H. W. Brinkmann* (Swarthmore, Pa.).

**Kawada, Yukiyosi. On the derivations in simple algebras.** Sci. Papers Coll. Gen. Ed. Univ. Tokyo **2**, 1–8 (1952).

The author's earlier study of derivations in number fields [Ann. of Math. (2) **54**, 302–314 (1951); these Rev. **13**, 324] is extended here to simple algebras. If $Q$ is any ring with an identity, $M$ a 2-sided unitary $Q$-module, and $R$ a subring of $Q$, the 1-dimensional cohomology group $H(Q, R, M)$ of $Q$ in $M$, relative to $R$, is defined as the group of the derivations of $Q$ into $M$ which vanish on $R$, modulo the subgroup of the inner derivations $q \to q\cdot m - m\cdot q$, with some $m \in M$. With $Q$ a maximal order of a central simple algebra $A$ over an algebraic number field $k$, $R$ the ring of integers in $k$, and $B$ a 2-sided ideal in $Q$, the author determines $H(Q, R, Q/B)$.

Using Hasse's classical results on the arithmetic of $A$ [Math. Ann. **104**, 495–534 (1931)], the problem is reduced to a simple local situation. First, the decomposition of $B$ into a product of prime powers induces a corresponding decomposition of $H(Q, R, Q/B)$. If $B$ is a prime power $P^r$, this group is isomorphic with the corresponding local group for the $P$-adic completion of $A$. This reduces the problem to the case of a central simple algebra over a $P$-adic field, which reduces further to the case of a $P$-adic division algebra. By using the representation of such an algebra as a crossed product of an unramified field extension, it is shown that $H(Q, R, Q/P^r)$ is (0) if $r$ is congruent to 1 modulo the local index $n_P$ of $A$, and is isomorphic with the additive group of the residue class field of $k$ at $P$ otherwise. From the known fact that the contribution of the prime ideal $P$ in $Q$ to the different of $Q$ is the $(n_P-1)$th power of $P$, one thus

has a connection between the groups $H(Q, R, Q/B)$ and the different of $Q$. *G. Hochschild* (Urbana, Ill.).

**Osima, Masaru. On the Schur relations for the representations of a Frobenius algebra.** J. Math. Soc. Japan 4, 1–13 (1952).

The author gives a new proof for the Schur-Nakayama relations for the coefficients of the regular representation of a Frobenius algebra $A$ [cf. Nakayama, Proc. Amer. Math. Soc. 3, 183–195 (1952), and the literature cited there; these Rev. 14, 240]. Some new relations of a similar type are given for the case that a Cartan basis of $A$ is used for forming the regular representation. A number of remarks concerning corresponding bases of a Frobenius algebra precede the proof. *R. Brauer* (Cambridge, Mass.).

**Šneĭdmyuller, V. I. Algebras of infinite rank with minimality condition for subalgebras.** Doklady Akad. Nauk SSSR (N.S.) 86, 665–668 (1952). (Russian)

This is a companion paper to an earlier one by the author [Mat. Sbornik N.S. 27(69), 219–228 (1950); these Rev. 12, 314] on rings with minimum condition for subrings. Let $A$ be an algebra with minimum condition for subalgebras. Then $A$ is algebraic and moreover its radical is finite-dimensional; this is a simplification as compared with the ring problem, for an example like the group of type $p^\infty$ is ruled out. On the other hand, the local finiteness which was easily established for rings remains an open question here. The main theorem asserts that $A$ is the direct sum of a finite-dimensional algebra and a finite number of division algebras. Moreover, if such a division algebra is locally finite, it is finite-dimensional over its center. *I. Kaplansky.*

**Herz, Jean-Claude. Sur les systèmes de polynomes différentiels.** C. R. Acad. Sci. Paris 235, 1085–1087 (1952).

After observing that the set of all sets of differential polynomials in indeterminates $y_1, \cdots, y_n$ over an ordinary differential field of characteristic 0 can be preordered by means of characteristic sets and that the corresponding ordered set is well-ordered, the author proves anew Ritt's basis theorem by induction in this well-ordered set.

*E. R. Kolchin* (New York, N. Y.).

## *Theory of Groups*

**Mal'cev, A. I. Symmetric groupoids.** Mat. Sbornik N.S. 31(73), 136–151 (1952). (Russian)

L'auteur appelle groupoïde un ensemble muni d'une multiplication partout définie. Un groupoïde $G$ sera dit symétrique s'il existe un ensemble $M$ tel que $G$ soit isomorphe au groupoïde (associatif) constitué par l'ensemble $C_M$ de toutes les applications de $M$ dans lui-même, la multiplication dans $C_M$ étant identique à la composition des applications.

La première partie de ce travail consiste à donner une caractérisation abstraite du concept de groupoïde symétrique: Soit $N$ l'ensemble des zéros à gauche de $G$ (c'est-à-dire, l'ensemble des éléments $n$ tels que, pour tout $g \in G$, $ng=n$) et soit $\tau_N$ l'ensemble des translations à gauche $\gamma_{n_0}=(n\to n_0n)$. Pour que $G$ soit un groupoïde symétrique, il faut et il suffit que $C_N\subset\tau_N$ et que $g_1n=g_2n$ pour tout $n\in N$ implique $g_1=g_2$. On déduit alors très facilement de cette proposition le théorème de Schreier: tout automorphisme d'un groupoïde symétrique est un automorphisme intérieur. La seconde partie de ce travail consiste à déterminer tous les groupoïdes quotients d'un groupoïde symétrique ou, ce qui revient au même, toutes les relations d'équivalence compatibles avec la structure multiplicative.

Si $M$ est fini, l'ensemble de toutes les relations d'équivalence compatibles avec la structure de groupoïde de $C_M$ est identique à l'ensemble des relations d'équivalence $R$ de la forme suivante: "$f\equiv g \bmod R$ si et seulement si, $n$ étant un entier inférieur ou égal au nombre d'éléments de $M$ et $\mathfrak{N}_n$ étant un sous-groupe distingué du groupe symétrique de degré $n$, l'une des deux conditions suivantes est remplie: (1) ou bien $f(M)$ et $g(M)$ ont chacun un nombre d'éléments inférieur à $n$; (2) ou bien $f(M)=g(M)$ a un nombre d'éléments égal à $n$, $f^{-1}f=g^{-1}g$ et $f\in\mathfrak{N}_ng$. Dans le cas où $M$ est infini, pour obtenir la description complète de toutes les équivalences compatibles il faut encore ajouter à l'ensemble des équivalences du type précédent un ensemble d'équivalences d'un autre type qu'il serait trop long de décrire ici.

Les résultats précédents permettent de retrouver immédiatement le résultat de Vorob'ev [Doklady Akad. Nauk SSSR (N.S.) 58, 1877–1879 (1947); ces Rev. 9, 330] suivant lequel quand $M$ est fini, les seuls sous-systèmes du groupoïde symétrique $C_M$ qui sont identiques au noyau d'un certain homomorphisme sont les sous-groupes distingués du groupe symétrique $\mathfrak{S}_M$ et de donner une description de ces mêmes sous-systèmes quand $M$ est infini. *J. Riguet* (Paris).

**Richert, Hans-Egon. Über die Anzahl Abelscher Gruppen gegebener Ordnung. I.** Math. Z. 56, 21–32 (1952).

Let $a_0(n)$ denote the number of essentially different abelian groups of order $n$ and put $A_0(x)=\sum_{n\le x}a_0(n)$. Then D. G. Kendall and the reviewer [Quart. J. Math., Oxford Ser. 18, 197–208 (1947); these Rev. 9, 226; for errata see review of a paper by Šapiro-Pyateckiĭ, Mat. Sbornik N.S. 26(68), 479–486 (1950); these Rev. 12, 316] showed that

$$A_0(x)=C_1x+C_2x^{1/2}+O(x^{1/3}\log^2 x)$$

where $C_1$ and $C_2$ are certain constants. This result was obtained by applying a famous theorem of Landau to the function $\zeta(s)\zeta(2s)$ which consists of the first two factors of the generating function

$$\sum_{n=1}^{\infty}a_0(n)n^{-s}=\prod_{\nu=1}^{\infty}\zeta(\nu s).$$

By dispensing with this theorem and applying instead van der Corput's method to estimate the sum-function of the coefficients of the first three factors $\zeta(s)\zeta(2s)\zeta(3s)$, the author obtains the sharper result

$$A_0(x)=C_1x+C_2x^{1/2}+C_3x^{1/3}+O(x^{3/10}\log^{9/10}x).$$

For this purpose estimates of the sums

$$D(x;\mu,\nu)=\sum_{m^\mu n^\nu\le x}1\quad(x\geqq1,\ \mu>\nu>0)$$

are needed and these involve estimates of sums of the type

$$\sum_{n\le x^\alpha}\left\{\frac{x^\beta}{n^\beta}-\left[\frac{x^\beta}{n^\beta}\right]-\frac{1}{2}\right\}.$$

These latter sums are essentially trigonometric sums and it is to them that van der Corput's theory of exponent-pairs, as simplified by Phillips [Quart. J. Math., Oxford Ser. (1) 4, 209–225 (1933)], is applied. The index 3/10 appears as a result of taking the exponent-pair (2/7, 4/7); as the author remarks, a minute diminution of the index could be achieved by making a different choice of exponent-pair. The reviewer confirms this as his calculations show that 3/10 exceeds the smallest index which the method can yield by less than 0.00007. *R. A. Rankin* (Birmingham).

**Nachbin, Leopoldo. On a duality theorem for commutative groups.** Anais Acad. Brasil. Ci. **24**, 137–142 (1952).

The author proves a theorem stated without proof by S. MacLane [Bull. Amer. Math. Soc. **56**, 485–516 (1950); these Rev. **14**, 133]. All groups considered are abelian. $Q$ is the additive group of rational numbers, $Z$ the additive group of integers. The author calls a group $G$ left separating (l.s.) if whenever $H_1$, $H_2$ are groups and $\alpha$, $\beta$ are homomorphisms $H_2 \to H_1$ with $\alpha \neq \beta$, there exists a homomorphism $\varphi: H_1 \to G$ such that $\varphi\alpha \neq \varphi\beta$. The theorem in question asserts that a group $G$ is l.s. if and only if there exists an isomorphism of $Q/Z$ into $G$. This property characterizes $Q/Z$. That is, if $H$ is a group such that the class of groups into which $H$ can be mapped isomorphically is just the class of l.s. groups, then $H \cong Q/Z$. *P. A. Smith* (New York, N. Y.).

**Hall, Marshall, Jr. A combinatorial problem on abelian groups.** Proc. Amer. Math. Soc. **3**, 584–587 (1952).

Soit $A$ un groupe abélien fini, noté additivement, et soit $\Phi$ une application de $A$ dans lui-même. L'auteur démontre le théorème: Pour que $\sum_{x \in A} \Phi(x) = 0$, il faut et il suffit qu'il existe deux permutations $\sigma$ et $\tau$ de $A$ telles que, quel que soit $x \in A$, $\sigma(x) - \tau(x) = \Phi(x)$. (La démonstration s'effectue par un processus d'itération conduisant à la solution au bout d'un nombre fini de pas.) En particulier, si $k$ est une application de $A$ dans l'ensemble des entiers positifs ou nuls, la condition nécessaire et suffisante pour qu'il existe deux permutations $\sigma$ et $\tau$ de $A$ telles que l'équation $\sigma(x) - \tau(x) = y$ admette $k(y)$ solutions en $x$ est que $k$ satisfasse aux deux conditions:

$$\sum_{x \in A} k(x) = \text{ordre de } A, \quad \sum_{x \in A} k(x) \cdot x = 0.$$

Plus particulièrement, la condition nécessaire et suffisante pour qu'il existe une permutation $\sigma$ de $A$ transversale (c'est-à-dire, telle que l'application $(x \to \sigma(x) - x)$ soit biunivoque) est que $A$ contienne plus d'un élément d'ordre 2 ou n'en contienne pas du tout, ce qui avait déjà été démontré par Lowell Paige dans sa thèse [Univ. of Wisconsin, 1947]. *J. Riguet* (Paris).

**Green, J. A. On groups with odd prime-power exponent.** J. London Math. Soc. **27**, 476–485 (1952).

Let $q$ be a power of the odd prime $p$, and also $q \neq 3$. Then there exists a finite group of exponent $q$, generated by two elements, which has class $2q-2$. Not only this result but the method of proof is of interest for the Burnside problem. Consider the ring $R$ of formal sums of monomials which are words in the indeterminates $x$ and $y$, with coefficients integers modulo $p$. In $R$ the elements $1+x$ and $1+y$ generate a group, which we call $F$. In the commutator $[[xy], [x^{q-1}y], [x^{q-2}y]]$ the words $A = xyx^{q-2}yx^{q-1}yx$ and $B = xyx^{q-1}yx^{q-2}yx$ occur with coefficients which are respectively 2 and $-2$. But he shows that $A$ and $B$ occur with the same coefficient in every $q$th power and product of $q$th powers. This he is able to show since in a $q$th power $(\sum a_i u_i)^q$ where the $u_i$ are monomials and the $a_i$ coefficients, the resulting polynomials in the $a$'s are commutative and hence more amenable to calculation. From these results, $F$ modulo $q$th powers and commutators of weight $2q-1$ will be a finite group whose existence was to be shown.
*Marshall Hall* (Columbus, Ohio).

**Clowes, J. S. On groups of odd order.** J. London Math. Soc. **27**, 507–510 (1952).

Theorem 1: If $H$ is a $p$-group of odd index in $G$, the number of Sylow groups of $G$ containing $H$ is odd. Theorem 2: If $H$ is of order dividing $m$, of odd index in $G$ where $G$ is solvable, and of order $mn$ with $(m, n) = 1$, the number of subgroups of $G$ of order $m$ containing $H$ is odd. The proof of Theorem 1, by triple induction, hinges upon counting, modulo 2, the Sylow groups containing $H$ according to their intersection with the normalizer of $H$. The proof of Theorem 2 is independent and more elementary. *R. C. Lyndon.*

**Grün, Otto. Über eine gewisse Klasse von endlichen Gruppen.** Math. Nachr. **8**, 167–169 (1952).

Let $p$ be a prime which divides the order of a group $G$. Then $G$ is called $p$-special if, for each non-trivial $g \in G$, the order of $g$ is relatively prime to $p$ or is a power of $p$. The group of isometries of a regular $p$-gon is a $p$-special group which is not a $p$-group. For $p$-special groups which are not $p$-groups the following are established: the normalizer of the center of a Sylow subgroup of order $p^k$ is the normalizer of the Sylow subgroup itself. The centralizer of a $p$-subgroup is a $p$-subgroup. If the center of a Sylow $p$-subgroup $P$ is included in a conjugate $Q$ of $P$, then the center of $Q$ is included in $P$. The question remains open as to whether there exist non-trivial intersections of Sylow $p$-subgroups. If such an intersection exists, then it includes the center of some one of its defining Sylow $p$-subgroups. If it is maximal with respect to being such an intersection, then it includes all such centers. If it is minimal with respect to being such an intersection, then it is conjugate to all the other minimal intersections, is a normal subgroup of a normalizer of some Sylow $p$-subgroup, and includes the center of this Sylow subgroup. *F. Haimo* (St. Louis, Mo ).

**Kemhadze, Š. S. On regularity of $p$-groups for $p=2$.** Soobščeniya Akad. Nauk Gruzin. SSR. **11**, 607–611 (1950). (Russian)

Every non-abelian 2-group is not regular in the sense of Philip Hall's definition of regularity for $p$-groups. It is first shown that a non-abelian $p$-group all of whose proper subgroups are abelian is regular if $p \neq 2$, but not if $p=2$. Since a subgroup of a regular group is regular, it follows by induction that the non-abelian 2-groups are not regular.
*Marshall Hall* (Columbus, Ohio).

**Čunihin, S. A. On subgroups of a finite group.** Doklady Akad. Nauk SSSR (N.S.) **86**, 27–30 (1952). (Russian)

In extending results due to Schur [S.-B. Preuss. Akad. Wiss. **1902**, 1013–1019], Hall [J. London Math. Soc. **3**, 98–105 (1928)], and Ore [Duke Math. J. **5**, 431–460 (1939), pp. 448–454], the author applies several of his own previous investigations [in particular, see same Doklady (N.S.) **59**, 443–445 (1948); **83**, 663–665 (1952); these Rev. **9**, 492; **13**, 818]. The definition of a $\Pi$-Sylow divisor of the order of a group is now extended to permit $\Pi$ to be vacuous. A group $\mathfrak{G}$ is $\Pi$-separable if and only if $\mathfrak{G}$ has a $\Pi$-permutable series, each of whose indices has among its distinct prime factors at most one prime belonging to $\Pi$. Thus if $b$ is the maximal $\Pi$-Sylow divisor of the order of $\mathfrak{G}$ and if $\mathfrak{G}$ has a $\Pi$-permutable series as described, then $\mathfrak{G}$ contains a solvable subgroup $\mathfrak{B}$ of order $b$ and all subgroups of order $b$ are conjugates of $\mathfrak{B}$. Let the finite group $\mathfrak{G}$ contain a normal subgroup $\mathfrak{G}_1$ of order $g_1$ and let $\mathfrak{G}/\mathfrak{G}_1$ have a subgroup of order $b$; let $\Pi$ be the collection of distinct prime factors of $b$ and let $b'$ be a divisor of $g_1$ which is divisible by the maximal $\Pi$-Sylow divisor of $g_1$; if $\mathfrak{G}_1$ contains a subgroup $\mathfrak{B}'$ of order $b'$ and if all subgroups of $\mathfrak{G}$ conjugate to $\mathfrak{B}'$ in $\mathfrak{G}$ are conjugate to $\mathfrak{B}'$ in $\mathfrak{G}_1$, then $\mathfrak{G}$ contains (at least one) subgroup of order $bb'$. A sufficient condition that the group $\mathfrak{G}$ of order $g$ be

Π-separable is the following: for every factorization of $g$ in the form $g=mn$, where $(m, n)=1$ and where $n$ is a Π-Sylow divisor of $g$, there exist subgroups of $\mathfrak{G}$ with respective orders $m$ and $n$. *R. A. Good* (College Park, Md.).

**Zavalo, S. T. Free operator groups.** Doklady Akad. Nauk SSSR (N.S.) **85**, 949–951 (1952). (Russian)

Let $S$ be a semigroup. If elements $x_i\alpha$, $i$ from a set of indices and $\alpha$ from $S$, are taken as free generators of a free group $G$, then $G$ is called a free operator group with $S$ as operators, where $(x_i\alpha)\beta=x_i(\alpha\beta)$ for generators and $(a_1a_2\cdots a_t)\sigma=a_1\sigma\cdot a_2\sigma\cdots a_t\sigma$ defines the effect of an operator on products of elements which are generators or inverses of generators. Under this definition every group with operators is a factor group of an operator-free group. A subgroup $U$ of an operator-free group will be an operator free group if it is assumed that $U$ is completely admissible. By completely admissible it is meant that for $g \in G$, $\alpha \in S$, both or neither of $g$ and $g\alpha$ belong to $U$. *Marshall Hall.*

**Neumann, B. H., and Neumann, Hanna. Extending partial endomorphisms of groups.** Proc. London Math. Soc. (3) **2**, 337–348 (1952).

If $\mu$ is a homomorphism mapping a subgroup $A$ of $G$ onto a subgroup $B$ of $G$, then $\mu$ is called a "partial endomorphism" of $G$. Moreover, if there exists an endomorphism $\mu^*$ of $G^*\supseteq G$ which coincides with $\mu$ on $A$, then $\mu^*$ is called an extension of $\mu$, and $\mu$ is said to be "totally extendible". The authors prove that the necessary and sufficient condition for $\mu$ to be totally extendible is the existence of an ascending sequence $L_1\subseteq L_2\subseteq\cdots$ of normal subgroups of $G$ with the following property: Putting $n=1, 2, \cdots, K_n=L_n\cap A$, then $K_1=K$ is the kernel of $\mu$ and $K_{n+1}\mu=L_n\cap B$. If $\mu$ is an isomorphism, then $1=K_1=L_1=K_2=\cdots$ and $\mu$ is totally extendible and $\mu^*$ is an automorphism. This special result was established in an earlier paper of G. Higman and the authors [J. London Math. Soc. **24**, 247–254 (1949); these Rev. **11**, 322]. Furthermore, $\mu$ is totally extendible when $A$ is an $E$-subgroup of $G$, i.e., when every normal subgroup of $A$ is the intersection of $A$ with a normal subgroup of $G$; hence every partial endomorphism of an Abelian group $G$ can be extended to an endomorphism of some $G^*\supseteq G$. In addition it is shown that in this case $G^*$ can be selected as an Abelian group. The principal tool used in the paper is the free product of groups with amalgamation of subgroups. The problem (proposed at the end of the paper ) whether $G^*$ may be selected as a finite group when $\mu$ is extendible and $G$ finite may need the help of different methods.
*F. W. Levi* (Berlin).

**Szele, T. On groups with atomic layers.** Acta Math. Acad. Sci. Hungar. **3**, 127–129 (1952). (Russian summary)

Let $G$ be a group, let $C$ be the group of rationals modulo 1 and, for each $m$ $(m=1, 2, 3, \cdots, \infty)$, let $L(m)$ be the set of all $x$ in $G$ with order $m$. The set $L(m)$ is called a layer of $G$ and is said to be atomic if $L(m)$ has no proper subset which is a layer of some subgroup of $G$. Then the following are proved to be equivalent: (a) If $A$ and $B$ are distinct subgroups of $G$, then $A$ and $B$ are not isomorphic; (b) every layer of $G$ is atomic; and (c) $G$ is isomorphic to some subgroup of $C$. *F. Haimo* (St. Louis, Mo.).

**McLean, David. Cubic equations in groups.** Amer. Math. Monthly **59**, 624–626 (1952).

For each natural number $n$ there is constructed a group in which all root extractions of degree $\leqq n$ have meaning, but such that there is a "cubic" equation $ua_1ua_2u=1$ with no solution $u$ in the group. *D. G. Higman.*

**Garrido, Jules. Les groupes de symétrie des ornements employés par les anciennes civilisations du Mexique.** C. R. Acad. Sci. Paris **235**, 1184–1186 (1952).

**Wallace, Andrew H. Invariant matrices and the Gordan-Capelli series.** Proc. London Math. Soc. (3) **2**, 98–127 (1952).

This paper gives a fresh slant to the classical theory of invariants and to the theory of representations of the full linear group. The author starts from the explicit development of the group algebra of the symmetric group given by Rutherford [Substitutional analysis, Edinburgh, 1948; these Rev. **10**, 280] and from Turnbull's development of the Gordan-Capelli series and standard forms [An introduction to the theory of canonical matrices, 2nd ed., Blackie, London, 1945], and gets a proof of the complete reducibility of any tensor representation of the full linear group. He continues with an explicit construction of an irreducible representation on tensors of rank $n$ for each partition of $n$. This construction uses the Clebsch-Aronhold symbolic methods and the representation obtained is a more explicit formulation of the one given by Littlewood [The theory of group characters, Oxford, 1940; these Rev. **2**, 3]. The character of this representation is expressed in terms of the Gordan-Capelli series.

In Part II the author reverses the process and calculates the Gordan-Capelli coefficients from the representations. The major tool used in this analysis is the utilization of invariant matrices of differential operators which are generalizations of Cayley's Ω-process and Capelli's operators and goes back to the work of Deruyts [Bull. Acad. Roy. Sci. Lett. Beaux-Arts Belgique (3) **32**, 82–94 (1896)].
*R. M. Thrall* (Ann Arbor, Mich.).

**Farahat, H. On $p$-quotients and star diagrams of the symmetric group.** Proc. Cambridge Philos. Soc. **48**, 737–740 (1952).

The equivalence, in essence, of the notions of $p$-quotients of Littlewood [Proc. Roy. Soc. London. Ser. A. **209**, 333–353 (1952); these Rev. **14**, 243] and star diagrams of Osima-Robinson-Staal [Staal, Canadian J. Math. **2**, 79–92 (1950); these Rev. **11**, 415] is shown. Robinson's formula on $p$-index in the degree of an irreducible representation of a symmetric group [Trans. Roy. Soc. Canada. Sect. III. (3) **41**, 20–25 (1947); these Rev. **10**, 678] is proved directly, without making use of the reviewer's formula in terms of $p$-series.
*T. Nakayama* (Nagoya).

**Osima, Masaru. On the induced characters of a group.** Proc. Japan Acad. **28**, 243–248 (1952).

In the first two sections, the author sketches a proof of a theorem of the reviewer on induced characters [Ann. of Math. (2) **48**, 502–514 (1947); these Rev. **8**, 503]. No details are given. The method seems to be essentially that of the original paper. The last section contains an outline of a proof of a related earlier result of the reviewer [ibid. **42**, 53–61 (1941); these Rev. **2**, 215] which deals with the Cartan invariants of modular group algebras.
*R. Brauer* (Cambridge, Mass.).

**Bauer, Friedrich L. Tenseurs, dont les éléments sont des matrices de Dirac.** C. R. Acad. Sci. Paris **235**, 793–794 (1952).

The author proves the theorem: La caractéristique de la représentation spinorielle $^{2\nu+1}O(m_1+\frac{1}{2}, m_2+\frac{1}{2}, \cdots, m_\nu+\frac{1}{2})$

($m_i$ entier) du groupe orthogonal de dimension $2v+1$, c'est-à-dire, la caractéristique de la représentation principale qui résulte de la fusion

$$^{2v+1}O(m_1, m_2, \cdots, m_v) \times {}^{2v+1}O(\tfrac{1}{2}, \tfrac{1}{2}, \cdots, \tfrac{1}{2})$$

d'une représentation tensorielle avec la représentation par des matrices généralisées de Dirac (algèbre de Clifford), est le produit des caractéristiques de la représentation ${}^{2v+1}O(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ et de la représentation ${}^{2v}\mathrm{Sp}(m_1, m_2, \cdots, m_v)$ du groupe simplectique en $2v$ dimensions. *A. H. Taub.*

**Inönü, E., and Wigner, E. P. Representations of the Galilei group.** Nuovo Cimento (9) 9, 705–718 (1952).

The irreducible unitary representations of the Galilean group (generated by space-time translations, space rotations, and transitions to uniformly moving coordinate systems) are determined by the method of induced representations used in Wigner's determination of the representations of the inhomogeneous Lorentz group. It is found that, unlike the case of ray representations, none of the present representations admits a satisfactory physical interpretation, along certain conventional lines at least. *I. E. Segal.*

**Gel'fand, I. M., and Naĭmark, M. A. Unitary representations of a unimodular group containing an identity representation of the unitary subgroup.** Trudy Moskov. Mat. Obšč. 1, 423–475 (1952). (Russian)

The authors determine all representations of "class 1" of the complex unimodular group $G$, such a representation being defined as a (continuous, unitary) one for which there exists a vector $x$ in the representation space whose transforms $T(a)x$ ($a \in G$) span the space, and which is invariant under the unitary subgroup $U$ of $G$, i. e., $T(u)x=x$ for $u \in U$. ($U$ is the maximal compact subgroup of $G$, unique within inner automorphisms.) The representations of class 1 of the principal and complementary series of irreducible representations, previously described by the authors, exhaust the representations of class 1. The proof involves showing that the subring $R_0$ of the group ring of $G$ consisting of functions $f$ satisfying the equation $f(au)=f(ua)=f(a)$ for all $u \in U$, is commutative, and that each representation of class 1 of $G$ corresponds to a symmetric maximal ideal of $R_0$. This correspondence involves the use of the elementary positive definite functions $\phi$ on $G$ which are invariant under left and right translations by elements of $u$; the above representation of class 1 yields such a function via the equation $\phi(a)=(T(a)x, x)$. Such functions $\phi$ are called "spherical functions of irreducible representations of class 1", and were previously treated by the authors for the case of the principal series [Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 239–260 (1950); these Rev. 12, 9]. Here formulas are given for the spherical functions in the case of the complementary and degenerate series.

It is stated that the present methods will be extended in later work to cover the case of representations which contain arbitrary fixed irreducible representations of the unitary subgroup (the case treated here being that when this is the identity representation). It is also stated that the methods extend to all the classical groups. *I. E. Segal.*

**Loomis, L. H. Note on a theorem of Mackey.** Duke Math. J. 19, 641–645 (1952).

The author presents a new proof of a theorem of G. W. Mackey [same J. 16, 313–326 (1949); these Rev. 11, 10]. Mackey's proof uses the theory of unitary equivalence of self-adjoint operators and is formulated under the assumption of separability (though the latter could conceivably be avoided). In the present paper the theorem is proved in such a manner as to avoid the above. It seems to the reviewer that the method of proof is of interest in itself and could perhaps be used advantageously in related problems.

*F. I. Mautner* (Baltimore, Md.).

**Tôyama, Hiraku. On Haar measure of some groups.** Proc. Japan Acad. 24, 13–16 (1948).

The volume element $d\Omega$ for Haar measure on the unitary group of order $n$ at the point $U$ is shown to be $d\Omega=|I+H^2|^{-n}dh$, where $H$ is the Cayley transform of $U$, and $dh$ is the volume element in the Euclidean $n^2$-dimensional space associated with the matrix for $H$. Similar formulas are computed for the volume elements of the unitary symplectic groups and the orthogonal groups.

*L. H. Loomis* (Cambridge, Mass.).

# NUMBER THEORY

Ordered

***Davenport, H. The higher arithmetic. An introduction to the theory of numbers.** Hutchinson's University Library, London; Longmans, Green and Co., Inc., New York, N. Y., 1952. viii+172 pp. Text Ed. $1.80; Trade Ed. $2.25.

This book is an extraordinarily lucid account of elementary number theory. Although a few of the common topics such as the Möbius inversion formula are omitted, there is more than ample compensation in the fresh style which enables the author to shed much light on topics of considerable depth that are here only touched on. Throughout the book one encounters the results and spirit of modern research as well as references to the literature. As a result, this work can serve not only as an account of elementary results but also as a guide to further reading. Unfortunately, the suitability of this book as a text is diminished by the fact that there are no problems.

Chapter I, Factorization and the Primes, deals with the algebraic properties of integers, induction, the unique factorization theorem (for which a proof by induction is given), the Euclidean algorithm and a general discussion of primes. Chapter II, Congruences, is concerned with the Fermat-Euler theorem, simultaneous congruences, Wilson's theorem, and algebraic congruences including Lagrange's theorem. The author also proves Chevalley's result that a polynomial congruence in $n$ variables of degree less than $n$ with 0 as its constant term always has a non-trivial solution modulo a prime. An explicit statement of the relationships between the congruences $f(x)\equiv 0\ (p^n)$ and $f(x)\equiv 0\ (p^{n-1})$ would clarify some of the remarks of Chapter VI and place these in their more general setting.

In Chapter III, Quadratic Residues, is proved the existence of a primitive root modulo a prime. Indices, $k$th power residues, Euler's criterion, Gauss' lemma, the quadratic reciprocity law and the number of pairs of consecutive quadratic residues or non-residues are treated as well.

Chapter IV, Continued Fractions, is rather long for a book of this kind. The usual recurrence relations and Euler's rule are proved. The convergence of the infinite simple continued fraction is proved as is the Diophantine approxima-

tion result that $|\alpha-x/y|<1/y^2$ for each convergent $x/y$ to the irrational number $\alpha$. Also proved is the fact that $\sqrt{N}$ can be developed into a periodic fraction with the usual symmetric property. The solutions of Pell's equation are now obtained and Klein's geometrical interpretation of continued fractions is given. The continued fraction expansions for $\sqrt{N}$ as well as the smallest integers $x$ and $y$ such that $|x^2-Ny^2|=1$ are listed for $N\leqq 50$.

Chapter V, Sums of Squares, proves the necessary and sufficient condition for a number to be the sum of two squares, states a number of constructions for the solutions $x$, $y$ of $x^2+y^2=p$ where $p$ is a prime of the form $4k+1$, proves Lagrange's theorem on the universality of the form $x^2+y^2+z^2+w^2$ and derives the necessary condition for an integer to be representable by the sum of three squares.

Chapter VI, Quadratic Forms, deals with the equivalence of binary forms and the representation of integers by positive definite binary forms. The reduction theory for positive definite forms is given although it is stated without proof that two reduced forms are not equivalent; the last result is used to prove the finiteness of the class number for positive definite forms. Some results on the number of proper representations are given; there is also a discussion of the Dirichlet class number formula for forms with fundamental discriminant equal to $-p$. A table of reduced positive definite forms of discriminant $-d\leqq 83$ is given.

Chapter VII, Some Diophantine Equations, obtains the general solution of $x^2+y^2=z^2$ and discusses both $ax^2+by^2=cz^2$ and the Fermat problem. The impossibility of $x^4+y^4=z^2$ is proved. This is followed by a discussion of $x^3+y^3=z^3+w^3$ and the Thue-Siegel theorem.

It is interesting to note that a factor contributing to the fine quality of the exposition is the author's rejection of the not uncommon practice of attempting to catch the reader's eye by frivolous anecdotes and number tricks. The result is a serious work of great charm. *L. Schoenfeld.*

***Stewart, B. M. Theory of numbers.** The Macmillan Company, New York, N. Y., 1952. xiii+261 pp. \$5.50.

This is an elementary textbook which gives some of the best known methods and theorems of the classical elementary number theory. In short chapters, to be presented in fifty-minute lectures, brief expositions on different subjects are given. The material is chosen in such a way that it may arouse the students interest in number theory, but it is loosely joined and the reader of the book will not find there so much of that systematical structure which characterizes modern number theory. *H. Bergström.*

**Moessner, Alfred. Eine Bemerkung über die Potenzen der natürlichen Zahlen.** S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 29 (1952).

**Perron, Oskar. Beweis des Moessnerschen Satzes.** S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 31–34 (1952).

The first of these papers states the following theorem, without proof: From the sequence of natural numbers $1, 2, 3, \cdots$ discard every $k$th one and form consecutively the sums of the numbers remaining; in the sequence thus obtained discard every $(k-1)$th number and form the sums of the numbers remaining; in the sequence thus formed discard every $(k-2)$th number and again form the sums; if this process is continued, the result of the $(k-1)$th step will be the sequence of $k$th powers: $1^k, 2^k, 3^k, \cdots$. For $k=2$ this is a well-known fact, but the generalization appears to be new. The proof is given in the second paper and consists in the derivation (by induction) of a formula for the result of the $r$th step ($r=1, 2, \cdots, k-1$) in the theorem. *H. W. Brinkmann* (Swarthmore, Pa.).

**Carletti, Ernesto. Definizione della forma $F_m^n$; sue proprietà elementari; e deduzione dei teoremi di Fermat, di Wilson, e di Staudt e Clausen.** Period. Mat. (4) 30, 153–159 (1952).

The writer proves the theorems indicated in the title by examining the function $F_m^n$ which represents all terms in the expansion of $(x_1+x_2+\cdots+x_r)^n$ which involve exactly $m$ of the indeterminates $x_i$. *I. Niven* (Eugene, Ore.).

**Skolem, Th. The general congruence of 4th degree modulo $p$, $p$ prime.** Norsk Mat. Tidsskr. 34, 73–80 (1952).

For a prime $p>4$ let $f(x)\equiv 0 \pmod p$ be a congruence of degree 4, $f$ having discriminant $D$ such that $p\nmid D$. The congruence is solved by use of theorems of the author [Norske Vid. Selsk. Forh., Trondhjem 10, 89–92 (1937)] for cubic congruences, the application here being to the resolvent cubic $g(x)$ of $f(x)$. Let $r_1$, $r_2$, $r_3$ denote the roots of $g(x)\equiv 0 \pmod p$ whenever they exist. There are five cases in the factoring of $f(x)$ modulo $p$: (1) 4 linear factors; (2) two irreducible quadratics; (3) one linear and one irreducible cubic; (4) two linear and one irreducible quadratic; (5) $f(x)$ irreducible. Necessary and sufficient conditions for these five cases are as follows: (1) the Legendre symbols $(D/p)=1$ and $(r_i/p)=1$ for $i=1, 2, 3$; (2) $(D/p)=1$, $(r_1/p)=1$, $(r_2/p)=(r_3/p)=-1$; (3) $(D/p)=1$ but no $r_i$ exists; (4) $(D/p)=-1$, $(r_1/p)=1$, $r_2$ and $r_3$ non-existent; (5) $(D/p)=-1$, $(r_1/p)=-1$, $r_2$ and $r_3$ non-existent. *I. Niven* (Eugene, Ore.).

**Brčić-Kostić, Mato. L'extension de la congruence**

$$(a+b)^n-a^n-b^n\equiv 0 \pmod n$$

**($n$ un nombre premier).** Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 7, 7–11 (1952). (Serbo-Croatian. French summary)

The author gives examples of the congruence

$$(1)\qquad (a+b)^n-a^n-b^n\equiv 0 \pmod{n^{2k+1}}$$

where $n$ is a prime of the form $6m+1$. The function $(x+1)^n-x^n-1$ and its derivative vanish when $x$ is a complex cube root of unity. Hence by the well-known divisibility property of the binomial coefficients, the form $(a+b)^n-a^n-b^n$ is divisible by $n(a^2+ab+b^2)^2$. Any power $n^k$ can be represented by the form $a^2+ab+b^2$. Hence the congruence (1) has solutions $(a, b)$ for every $k$. Five examples are given. *D. H. Lehmer* (Los Angeles, Calif.).

***Hanneken, Clemens Bernard. Irreducible quintic congruences.** Abstract of a thesis, University of Illinois, Urbana, Ill., 1952. 7 pp.

**Ljunggren, Wilhelm. New solution of a problem proposed by E. Lucas.** Norsk Mat. Tidsskr. 34, 65–72 (1952).

G. N. Watson [Messenger of Math. 48, 1–22 (1918)] proved, by use of elliptic functions, that the only integral solutions of $\sum_{j=1}^n j^2=q^2$ are $n=q=1$ and $n=24$, $q=70$, thus settling a question proposed by E. Lucas [Nouvelles Ann. Math. (2) 14, 336 (1875)]. The present paper gives a proof employing only arithmetic considerations, and sketches two other such proofs. The study of the units of certain biquadratic fields is involved. *I. Niven* (Eugene, Ore.).

**Stolt, Bengt. On the Diophantine equation $u^2-Dv^2=\pm 4N$. II.** Ark. Mat. **2**, 251–268 (1952).

Continuing his previous work [Ark. Mat. **2**, 1–23 (1952); these Rev. **14**, 247] the author now determines an upper bound for the number of classes of solutions of the equation of the title for general $N$, where in the earlier work he treated the cases with $N$ square-free. The detailed results are too lengthy for summary here. It is also established that for fixed $N$ the equation has at most one ambiguous class, that is, a class which contains the solution $(u, -v)$ whenever it contains $(u, v)$. *I. Niven* (Eugene, Ore.).

***Hemer, Ove. On the Diophantine equation $y^2-k=x^3$.** Thesis, University of Uppsala, Almqvist & Wiksells, Uppsala, 1952. 101 pp.

The first chapter contains results on binary cubic forms and cubic rings, and the correspondence between (1) $y^2-kf^2=x^3$ and binary cubic forms. The binary cubic forms arise from equating irrational parts in

$$\pm y+f\sqrt{k}=\epsilon(a+b\sqrt{k})^3,$$

$\epsilon$ being a unit in $R(\sqrt{k})$, in order to solve (1). There are special theorems giving bounds for the number of solutions $(x, y)$ of (1) in terms of the number of prime factors of $f$, under suitable hypotheses on $k$, $f$, and the class number of $R(\sqrt{k})$. Extensions and refinements are made to the theory of binary cubic forms with negative discriminant of B. Delaunay and T. Nagell; this theory relates the forms to fundamental units of cubic fields, and has been summarized by Nagell [L'analyse indéterminée de degré supérieur, Gauthier-Villars, Paris, 1929].

Applications are made in the second chapter to 37 equations with $0<kf^2\leqq 100$ which yield to the various methods and criteria of Chapter I, complete solutions being obtained. The other cases satisfying this inequality are treated, but for these the solutions obtained are not shown to be complete. Seven equations with $0<-kf^2\leqq 100$ are treated. In Chapter III the writer uses results of Billing [Nova Acta Soc. Sci. Upsaliensis (4) **11**, no. 1 (1938)], Cassels [Acta Math. **82**, 243–273 (1950); these Rev. **12**, 11], and Fueter [Comment. Math. Helv. **2**, 69–89 (1930)] to determine all solutions of 89 of the 100 equations $y^2+27k=x^3$ with $0<|k|\leqq 50$. Tables of solutions and fundamental units of the cubic fields are given. An extensive bibliography can be obtained from the works of Cassels and Nagell cited above, and from Th. Skolem [Diophantische Gleichungen, Springer, Berlin, 1938]. *I. Niven* (Eugene, Ore.).

**Duarte, F. J. General solution of a Diophantine equation of the third degree.** Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. **14**, no. 45, 3–7 (1951). (Spanish)

With arbitrary integral parameters $a$, $b$, $c$ the formulas

$$x, y=(a^2+3b^2)^2+(a\pm 3b)c^3,$$
$$z=(a^2+3b^2)^2-2ac^3, \quad v=3(a^2+3b^2)c^2$$

give solutions of $x^3+y^3+z^3-3xyz=v^3$. Conversely, to any integral solution $x_1$, $y_1$, $z_1$, $v_1$ of this equation there correspond integers $a$, $b$, $c$ giving values $x$, $y$, $z$, $v$ in the formulas such that $x/x_1=y/y_1=z/z_1=v/v_1$. *I. Niven.*

***Häggmark, Per. On a class of quintic Diophantine equations in two unknowns.** Thesis, University of Uppsala, Almqvist & Wiksells, Uppsala, 1952. 91 pp.

It is proved that (1) $u^5+mv^5=1$ has at most two integral solutions with $v\neq 0$, $m$ being any rational integer. For $|m|\geqq 2060$ it has been established by C. L. Siegel [Math. Ann. **114**, 57–68 (1937)] that this equation, among others, has at most one solution. The author uses a $p$-adic method developed by Skolem in a series of four papers [we cite the last: Math. Ann. **111**, 399–424 (1935)]. Each solution of (1) gives a corresponding unit $u+v\sqrt[5]{m}$ of $R(\sqrt[5]{m})$. It is established that if (1) has two non-trivial solutions, then the corresponding units form a system of independent units of $R(\sqrt[5]{m})$, and this result is employed in proving the central theorem. *I. Niven* (Eugene, Ore.).

**Georgiev, G. Résolution de l'équation**

$$\sum_{k=1}^{n} A_k \prod_{i=1}^{n} x_i^{a_{ki}} = A_0$$

**en nombres rationnels.** Acta Math. Acad. Sci. Hungar. **2**, 229–246 (1951). (French. Russian summary)

The equation of the title, with rational coefficients $A_k$ and integral exponents $a_{ki}$ with determinant $|a_{ki}|\neq 0$, is studied by use of the transformation (1) $x_i=\prod_{r=1}^{n} X_r^{\lambda_{ri}}$, $i=1, 2, \cdots, n$, with integral exponents having determinant $|\lambda_{ri}|=\pm 1$. Necessary and sufficient conditions that a transformation (1) exist which reduces the original equation to $\sum_{k=1}^{n} A_k X_k = A_0$ are that $|a_{ki}|=\pm 1$. Necessary and sufficient conditions are also given for reduction to $\sum A_k X_k^{m_k} = A_0$. These results lead to sufficient conditions for solving the equation in rational integers. In case one or more of the coefficients $A_k$ are zero, stronger results are obtained. *I. Niven* (Eugene, Ore.).

**Samet, P. A. An equation in Gaussian integers.** Amer. Math. Monthly **59**, 448–452 (1952).

This paper is concerned with the following extension of Legendre's Theorem: A necessary and sufficient condition for the equation $ax^2+by^2+cz^2=0$ to be soluble in Gaussian integers, not all zero, if $a$, $b$, $c$ are square-free Gaussian integers and are co-prime in pairs, is that $bc$, $ca$, and $ab$ be quadratic residues of $a$, $b$, and $c$ respectively. The main object is to show that the proof of Legendre's Theorem given in Dickson's "Modern elementary theory of numbers" [Univ. of Chicago Press, 1939, p. 155 ff.; these Rev. **1**, 65] can be adapted to the case where the coefficients and unknowns are Gaussian integers. *B. N. Moyls.*

**Ankeny, Nesmith C. The insolubility of sets of Diophantine equations in the rational numbers.** Proc. Nat. Acad. Sci. U. S. A. **38**, 880–884 (1952).

Denote by $M(T, a_1, a_2, \cdots, a_n)=M(T)$ the set of all positive integers $m<T$ for which the equation

$$(*) \qquad a_1X_1^m+a_2X_2^m+\cdots+a_nX_n^m=0$$

has a non-trivial solution in the rational integers. (A non-trivial solution is one in which not all of the $X$'s are zero.) Theorem I: If $\sum_{j=1}^{n} a_j\epsilon_j\neq 0$, where $\epsilon_j=\pm 1$ or 0, then $M(T)=o(T)$ as $T\rightarrow\infty$, i.e., for almost all $m$ equation (*) has no non-trivial solution. Theorem I excludes the important case $n=3$, $a_1=a_2=a_3=1$. However, some information can be gathered but only in the "first case" of Fermat's theorem. Theorem II: The density of integers $m<T$ for which the equation $X_1^m+X_2^m+X_3^m=0$ has a rational solution and for which $(X_1X_2X_3, m)=1$ is $o(T)$ as $T\rightarrow\infty$. The author sketches the proofs of a number of lemmas from which Theorems I and II follow. A main lemma states that if $a_1, a_2, \cdots, a_n$ satisfy the conditions of Theorem II, then for a given $m$ there exist no non-trivial rational solutions of (*) provided we can find a rational prime $p\not\equiv 1 \pmod{n!}$ such that $m|p-1$, $mr=p-1$, $(r, n!)=1$ or 2, $\phi(r)<\alpha^{-1}\log p$, where $\alpha=\log(|a_1|+|a_2|+\cdots+|a_n|)$, and $\phi(r)$ is the

Euler function. The proof is based upon ideas of algebraic number theory. There are also two lemmas which use tools of analytic number theory, namely the Brun-Selberg sieve methods and certain "probability" theorems regarding the density of numbers having a fixed number of prime factors.

*A. L. Whiteman* (Princeton, N. J.).

**Sierpiński, Waclaw. Sur une formule donnant tous les nombres premiers.** C. R. Acad. Sci. Paris 235, 1078–1079 (1952).

A real number $a$ is exhibited so that the $n$th prime $p_n=[af(n)]-f(n-1)[af(n-1)]$ where $[x]$ denotes the integral part of $x$ and $f(n)=10^{2^n}$. *I. Niven* (Eugene, Ore.).

**Nagura, Jitsuro. On the interval containing at least one prime number.** Proc. Japan Acad. 28, 177–181 (1952).

A refinement of the Ramanujan proof of "Bertrand's postulate" enables the author to show that for $x\geqq 25$ there is always a prime between $x$ and $1.2x$. The proof is based on de Polignac's identity

$$\sum_{m=1}^{\infty}\psi(x/m)=[x]!,\quad \psi(x)=\sum_{p^n\leqq x}\log p.$$

Using an integral representation of the logarithmic derivative of the Gamma function together with Sterling's approximation, the following inequalities are obtained:

$$.916x-2.318<\psi(x)<1.086x.$$

This is sufficient to prove that $\theta(6x/5)-\theta(x)>0$ for $x>2103$, where $\theta(y)=\sum_{p\leqq y}\log p$. Reference to the list of primes below 2103 establishes the above mentioned result.

*D. H. Lehmer* (Los Angeles, Calif.).

**Knödel, Walter. Sätze über Primzahlen. II.** Monatsh. Math. 56, 137–143 (1952).

Let $k$ be a fixed integer greater than unity and $y_1, y_2, \cdots, y_k$ a set of integers satisfying $y_1=0$, $0<y_i<2k\log x$ $(1<i\leqq k)$. Such a set is said to possess the property (E) if the $k$ numbers $s+y_i$ are all prime for more than $c_1(k)x/\log^k x$ values of $s$ less than $x$, where $c_1(k)$ depends only on $k$. The number of sets $(y_i)$ with property (E) is denoted by $Z$. In his first paper under the same title [Monatsh. Math. 55, 62–75 (1951); these Rev. 12, 676; in this review the condition $y_i<2\log x$ should be replaced by $y_i<2k\log x$] the author obtained lower and upper bounds for $Z$; in the present paper he obtains a better lower bound which is of the same order of magnitude as his earlier upper bound, namely

$$c_4(k)\log^{k-1}x<Z<c_6(k)\log^{k-1}x.$$

The proof is based on results obtained in the earlier paper and involves an estimate of the number of sets $(y_i)$ for which $\prod_{i>j}(y_i-y_j)$ is divisible by any integer $d$. The proof holds under the assumption $c_{12}>c_1c_6$ which is made without comment or verification. In the proof of Lemma 2 the author's deduction of the inequality (7) is hard to follow because of misprints or minor errors; it is, however, easily verified that (7) holds, and, in fact, remains true if $2k$ is replaced by $k$ in the product. *R. A. Rankin.*

**Wright, E. M. Functional inequalities in the elementary theory of primes.** Duke Math. J. 19, 695–704 (1952).

In this paper the author analyzes the functional inequalities behind Selberg's proof of the prime number theorem. Assuming real functions $\chi(x)>0$, $f(x)$, given, which are bounded and integrable in every finite interval $a\leqq x\leqq X$, and such that

$$(1)\quad |f(x_2)-f(x_1)|\leqq A_1|x_2-x_1|+A_2x_1^{-\delta},\quad A_1>0,\ A_2\geqq 0,\ \delta>0,$$

$$(2)\quad \left|\int_{x_1}^{x_2}f(y)dy\right|\leqq A_3,\quad A_3>0,$$

$$(3)\quad x|f(x)|\leqq\int_a^x|f(y)|dy+\chi(x);$$

the problem considered concerns the manner in which the order of $f(x)$ depends on the order of $\chi(x)$. The main theorem is the following. Theorem I: Under the above assumptions, if $\chi(x)=O(x\phi^2)$, where $\phi=\phi(x)>0$ is nonincreasing and (a) $\lim_{x\to\infty}\phi(x)=0$, (b) $\phi(x)(\log x)^{1/2}$ nondecreasing, then $f(x)=O(\phi(x))$ as $x\to\infty$. The special case where all that is known is that $\chi(x)=o(x)$ yields $f(x)=o(1)$, which is the relevant result for the proof of the prime number theorem (no error term). It is proved that Theorem I is best possible in the sense of Theorem II: If $\phi(x)$ satisfies the conditions of Theorem I, and $\chi(x)=x\phi^2$, there exists an $f(x)$ satisfying (1), (2), and (3) such that $f(x)\neq o(\phi)$ as $x\to\infty$.

*H. N. Shapiro* (New York, N. Y.).

**✶Linnik, Yu. V. Prime numbers and powers of two.** Trudy Mat. Inst. Steklov., v. 38, pp. 152–169. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

The author proves that the number of representations of an integer $N$ as a sum of two primes and $k$ powers of 2 is greater than

$$N\left(\frac{\log N}{\log 2}\right)^{k-1}(c_1-c_2(1-\eta)^{k-2})$$

where $c_1$, $c_2$ and $\eta<1$ are absolute constants and $k\geqq 3$. It follows at once that there exists a constant $k\geqq 3$ such that every large integer $N$ is representable in the form

$$N=p_1+p_2+2^{x_1}+\cdots+2^{x_k},$$

where $p_1$ and $p_2$ are primes and $x_1, \cdots, x_k$ are positive integers.

The proof is based on the investigation of the integral

$$\int_0^1 e(2\pi iN\alpha)S^2(\alpha)T^k(\alpha)d\alpha,$$

where

$$S(\alpha)=\sum_{p<N}e(-pN^{-1}-2\pi ip\alpha),$$

$$T(\alpha)=\sum_{m=1}^{\infty}e(-2^mN^{-1}-2\pi i2^m\alpha).$$

On the "major arcs" one can evaluate the integral

$$\int e(2\pi iN\alpha)S^2(\alpha)d\alpha,$$

whereas on the "minor arcs" one uses the inequality

$$\int_0^1|S(\alpha)|^2|T(\alpha)|^2d\alpha=O(N)$$

by Romanov [Math. Ann. 109, 668–678 (1934)] and a detailed ingenious study of the set of values of $\alpha$ where $T(\alpha)\geqq(1-\alpha)T(0)$. There is one serious misprint. The last displayed formula in the enunciation of the theorem on p. 154 should read

$$\sum_{(N_1)}>c_3N\left(\frac{\log N}{\log 2}\right)^{k-1}.$$

*H. Heilbronn* (Bristol).

**Obláth, R. Sur le problème de Goldbach.** Mathesis 61, 179–183 (1952).

The author supplements a note by Teghem [Mathesis 61, 45–47 (1952)] describing recent developments in Goldbach's problem.

**Sklar, Abe. On the factorization of squarefree integers.** Proc. Amer. Math. Soc. 3, 701–705 (1952).

The problem of "Factorisatio numerorum" is concerned with the number $f(n)$ of representations of an integer $n$ as an ordered product of factors greater than 1; it has been studied by Kalmár [Acta Litt. Sci. Szeged 5, 95–107 (1931)], Hille [Acta Arith. 2, 134–144 (1936)], Erdös [Ann. of Math. (2) 42, 989–993 (1941); these Rev. 3, 165], Sen [Bull. Calcutta Math. Soc. 33, 1–8 (1941); these Rev. 3, 269], and others. The author defines, for integral $r$, $h(r)=f(S_r)$, where $S_r$ represents any square-free integer with $r$ prime factors, and proves, among other results, a (convergent) asymptotic expansion:

$$h(r)=2^{-1}r!(\log 2)^{-r-1}\{1+R_r\},$$

where the remainder $R_r$ is expressed in an infinite series, involving two products of properly defined cosine terms. Using this result he determines a "normal" order of $f(n)$ for all $n$.

*S. Ikehara* (Tokyo).

**Cugiani, Marco. Sulla rappresentazione degli interi come somme di una potenza e di un numero libero da potenze.** Ann. Mat. Pura Appl. (4) 33, 135–143 (1952).

Theorem A: If $g\geqq 2$ is a fixed integer and $t=g$, then a number $N_0$ can be determined such that every positive integer $N\geqq N_0$ can be represented in the form $N=x^g+l_t$, where $(x, N)=1$ and $l_t$ is a $t$th power-free integer. Let $E(N)$ denote the number of such representations. Then there exist positive constants $\gamma_1, \gamma_2, \gamma_3$ such that

$$\text{(a)}\qquad E(N)>\gamma_1 N^{1/g}(\log\log N)^{-1}$$

for $N\geqq N_0$; (b) $E(N)<\gamma_2 N^{1/g}(\log\log N)^{-1}$ for infinitely many values of $N$; (c) $E(N)>\gamma_3 N^{1/g}$ for infinitely many values of $N$. Theorem B: Let $F(x)$ be an integral-valued polynomial of degree $g\geqq 2$ and put $t\geqq g$. Then a number $N_0$ can be determined such that every $N\geqq N_0$ can be represented in the form $N=F(x)+l_t$ with $F(x)>0$. Let $E(N)$ denote the number of such representations. Then there exists a positive constant $\gamma_4$ such that $E(N)>\gamma_4 N^{1/g}(\log\log N)^{1-g}$ for $N\geqq N_0$. Theorems A and B generalize in various directions the theorems of a number of writers among whom the following two may be cited: Rao [Proc. Indian Acad. Sci., Sect. A. 11, 429–436 (1940); these Rev. 2, 42] and Roth [J. London Math. Soc. 22, 231–237 (1947); these Rev. 9, 499]. The author's methods are related to those used by Ricci [Tôhoku Math. J. 41, 20–26 (1935)] to treat certain representation problems in additive number theory.

*A. L. Whiteman* (Princeton, N. J.).

**Bellman, Richard, and Shapiro, Harold N. On the normal order of arithmetic functions.** Proc. Nat. Acad. Sci. U. S. A. 38, 884–886 (1952).

Given an arithmetic function $f(n)$ the normal order of $F(x, y)=\sum f(n)$ taken over the range $x\leqq n\leqq x+y$ is defined to be $g(x)$ if for any $\epsilon>0$, we have

$$(1-\epsilon)g(x)\leqq F(x, y)\leqq(1+\epsilon)g(x)$$

for all $x\leqq z$ except for a set of total number $o(z)$ as $z\to\infty$. In this note a technique for deriving the normal order of $F(x, y)$ for a large class of arithmetical functions is indicated. The method employed is intimately connected to the methods introduced by Schnirelman [see E. Landau, Über einige neuere Fortschritte der additiven Zahlentheorie, Cambridge Univ. Press, 1937, pp. 40–59] and Turán [J. London Math. Soc. 9, 274–276 (1934)]. The authors apply their results to the problem of the distribution of squarefree numbers and prove: (I) If $y(x)$ is any function such that $\lim_{x\to\infty} y(x)=\infty$, then the number of $n\leqq z$ such that the interval $(n, n+y(n))$ contains no squarefree integer is $o(z)$ as $z\to\infty$. (II) If $\lim_{x\to\infty} y(x)=\infty$, and for some $\lambda$, $0<\lambda<1$, $y(x)/y(\lambda x)$ is bounded, then the normal order of $\sum\mu^2(n)$ taken over the range $x\leqq n\leqq x+y$ is $(6/\pi^2)y$. Here $\mu(n)$ is the Möbius function.

*A. L. Whiteman.*

**Sandham, H. F. Two identities due to Ramanujan.** Quart. J. Math., Oxford Ser. (2) 3, 179–182 (1952).

The author proves two identities stated by Ramanujan and uses them, together with the methods of a previous paper [J. London Math. Soc. 27, 107–115 (1952); these Rev. 13, 536], to deduce the partition series

$$\text{(1)}\qquad \sum_0^\infty \frac{p(n)}{2\cosh\frac{1}{6}\pi\sqrt{(24n-1)}+1}=\frac{\log(\sqrt{2}+1)}{\sqrt{3}},$$

$$\text{(2)}\qquad \sum_1^\infty \frac{p(n)}{2\cosh\frac{1}{6}\pi\sqrt{(24n-1)}-1}=\frac{\sqrt{3}}{2\pi}-\frac{1}{6}-\frac{\log 2}{\sqrt{3}}.$$

*N. J. Fine* (Philadelphia, Pa.).

**Iwasawa, Kenkichi, and Tamagawa, Tsuneo. Correction: On the paper "On the group of automorphisms of a function field".** J. Math. Soc. Japan 4, 203–204 (1952).

See same J. 3, 137–147 (1951); these Rev. 13, 325.

**Deuring, Max. Die Struktur der elliptischen Funktionenkörper und die Klassenkörper der imaginären quadratischen Zahlkörper.** Math. Ann. 124, 393–426 (1952).

This paper deals with a more or less complete arithmetic treatment of the celebrated theory of "complex multiplication", that is, the description of generating elements of abelian extensions $\Omega$ of an imaginary quadratic field $\Sigma$ as "singular modules" of the modular $j$-function and "division values" of elliptic functions whose arguments are related to the bases of well-defined ideals in rings of $\Sigma$. The author rebuilds the approach to complex multiplication in the manner of one of Hasse's earlier papers [J. Reine Angew. Math. 157, 115–139 (1926)] in which the proof of the law of reciprocity for $\Omega/\Sigma$ is a consequence of the theory of elliptic fields. The essential components of the author's theory are the following. (I) The theory of invariants of elliptic fields $K$ over coefficient fields $k$ of arbitrary characteristic, as presented in the author's paper (1) [Math. Z. 47, 47–56 (1940); these Rev. 3, 266]. There it is shown that elliptic fields $K/k$ and $K_1/k$ whose extensions by the algebraic completion of $k$ coincide are uniquely determined by a quantity $j$, the so-called modular invariant which has for characteristic distinct from 2 and 3 the classical form $j=2^6 3^3 g_2^3/\Delta$, where $y^2=4x^3-g_2x-g_3$, $\Delta=g_2^3-27g_3^2\neq 0$, is the Weierstrass defining equation of $K$. Here it is important that there exists for each value of $j$ an elliptic field $K$ whose smallest possible coefficient field equals $\mathrm{P}(j)$, $\mathrm{P}$ the prime field of the given field $k$. (II) The theory of coefficient reduction as started by the author in (2) [Math. Z. 47, 643–654 (1942); these Rev. 7, 362] and (3) [Abh. Math. Sem. Hansischen Univ. 14, 197–272 (1941); these Rev. 3, 104]. Briefly this entails the following: Suppose that $k$ is an algebraic number field and let $\mathfrak{p}$ be a prime ideal of $k$, then

the residue class mapping $k \to \{k/\mathbf{p}, \infty\}$ can be extended to a residue class mapping of the elliptic field, for almost all $\mathbf{p}$, so that the residue class field $\bar{K}$ is an elliptic field with the coefficient field $k/\mathbf{p}$ and the invariant $\bar{j} = j \bmod \mathbf{p}$. Furthermore the ring of multiplications $\mathbf{R}$ (meromorphisms, i.e., isomorphisms of $K/k$ into itself which are the identity on $k$) is mapped isomorphically into the ring of multiplications $\bar{\mathbf{R}}$ of $\bar{K}$ (see §4 of (3)). (III) The theory of elliptic subfields of $K/k$ (theory of division), for example in (3) loc. cit., stating that each elliptic subfield $K_0 \subset K$ is a join $K^{\mathfrak{a}} = \bigcup_{\mu \in \mathfrak{a}} K^{\mu}$, $\mathfrak{a}$ an ideal in $\mathbf{R}$ (a left ideal if $\mathbf{R}$ is a ring of quaternions). In the present paper the author reexamines and extends his former results of the division theory from the viewpoint of module theory with respect to $\mathbf{R}$ and in connection with the reduction theory. This is a preliminary step in the direction of the desired law of reciprocity. Furthermore, as a combination of results in (1) and (2), it follows that the invariants $j$ and $j_0$ of $K$ and $K_0$, respectively, satisfy the "invariant" equation $F(j, j_0) = 0$ with highest coefficients in $j$ and $j_0$ powers equaling $\pm 1$, and integral coefficients for characteristic 0, if $j$ is absolutely transcendental. (This furnishes the algebraization of the classical principle of "$q$-expansions".) Furthermore we find the crucial result that singular invariants $j$, i.e., those whose corresponding elliptic fields $K$ have multiplication rings $\mathbf{R}$ which are orders in imaginary quadratic fields $\Sigma$, are absolutely algebraic over the respective prime field of $K$. This means, in particular, for characteristic 0, that $F(j, j) = 0$, $F$ as above for a suitable $K_0 \subset K$. Finally there exist in $K$ as many non-isomorphic elliptic subfields $K_0$ as there are ideal classes with respect to $\mathbf{R}$ (§10 loc. cit. (3)). Basic for the theory of the author is a heretofore unexplored concept, namely, that of the "generalized meromorphism" $\Phi$ of an elliptic field $K/k$, that is, an isomorphism of $K$ into itself which maps $k$ not necessarily trivially upon itself, i.e., induces a possibly nontrivial isomorphism $\varphi$ on $k$. These generalized meromorphisms (playing later the role of symbolic generators of the ideals $\mathfrak{a} \subset \mathbf{R}$ (page 410) and referring to the proof of the principal ideal theorem for imaginary quadratic fields) are of utmost significance. It is shown in §4 how a given isomorphism $\varphi$ of $k$ can be extended to a generalized meromorphism $\Phi$ of $K$; although the concept is of algebraic nature, the whole theory of invariants (see I and III above) is used to secure the existence theorem. If the automorphism $\varphi$ of $k$ has finite order and has the fixed field $k_0$, then there exists a finite normal extension $k_1/k$ such that a prolongation of $\varphi$ to an isomorphism $\varphi_1$ of $k_1$ can be extended to a general meromorphism $\Phi_1$ of $Kk_1$. The proof involves that the conjugate $j^{\varphi}$ of the invariant $j$ of $K$ (it is assumed that $\mathbf{R}$ is imaginary quadratic) determines an elliptic field $K^{\mathfrak{a}}$, $\mathfrak{a}$ an ideal in $\mathbf{R}$ (see III above) which coincides after coefficient extension by the algebraic completion $\hat{k}$ of $k$ with the field $K^{\varphi}$ defined by $f^{\varphi}(X, Y) = 0$, $\varphi$ being applied to the coefficients of a suitable defining equation $f(x, y) = 0$ of $K$. Then $K^{\varphi}\hat{k}$ is isomorphic with $K^{\mathfrak{a}}\hat{k}$ and hence $K^{\mathfrak{a}}\hat{k}$ can be defined by $f^{\varphi}(x', y') = 0$, and one can set $x^{\Phi_1} = x'$, $y^{\Phi_1} = y'$ as desired. In addition, it is shown precisely how many extensions $\Phi_1$ exist, leading to the fact that each automorphism $\varphi$ of $k$ determines uniquely an ideal class with respect to $\mathbf{R}$, to wit, the class of $\mathfrak{a}$. On page 412 appears the statement that the Frobenius mapping $z \to z^p = z^{\Pi}$ on a field $K/k$ with absolutely algebraic coefficient field of characteristic $p \neq 0$ is a meromorphism which commutes with all multiplications and all generalized meromorphisms so that the corresponding ideal $\mathfrak{a}$ of $\mathbf{R}$ is prime. Furthermore, the reduction theory II is extended with the result that the prolongation $\mathbf{P}^*$ of a discrete rank-one valuation $\mathbf{P}$ of $k$ satisfies $\mathbf{P}^{*\Phi} = \mathbf{P}^*$ for an extension $\Phi$ of $\varphi$ to $K$ provided $\mathbf{P}^{\varphi} = \mathbf{P}$. The proof of the principal theorem of complex multiplication unfolds now as follows in the simplest case for which $\mathbf{R}$ is the maximal order in the given imaginary quadratic field $\Sigma$. Start with an elliptic field $K$ of characteristic 0 with the multiplication ring $\mathbf{R}$, its field of definition is to be $\Sigma = \Omega(j)$, $j$ the invariant of $K$ (see III above). Then the distinct nonisomorphic subfields of $K$ have invariants which are in 1-1 correspondence with the ideal classes $\mathfrak{k}$ of $R$. Now let $\varphi$ be an automorphism of the Galois group $G(\Omega/\Sigma)$ ($\Omega$ is recognized as a normal extension of $\Sigma$ as a consequence of the theory of the invariant and the fact that each conjugate of $j$ is again a suitable class invariant $j(\mathfrak{k})$). Then $\varphi$ determines a generalized meromorphism $\Phi_1$ on a suitable extension $K\Omega_1$, $\Omega_1$ normal over $\Sigma$; $\Phi_1$ in turn determines (see III) an ideal $\mathfrak{a}$ of $\mathbf{R}$, which in turn determines the ideal class $\mathfrak{k}$. Since distinct prolongations $\Phi_1$ of the same $\varphi$ give ideals in the same class $\mathfrak{k}$, the mapping $\varphi \to \mathfrak{k}(\varphi)$ is single-valued and (using the effect of the meromorphism $\Phi_1$ on the multiplication ring §4 #4) it follows that the mapping is an isomorphism of $G(\Sigma/\Omega)$ into the class group of $\Sigma$ with respect to $\mathbf{R}$. Next, to obtain the full law of reciprocity, the reduction theory (II and its extension to generalized meromorphisms) is used to prove that, given a prime ideal $\mathbf{p}$ of $\Sigma$ with the Frobenius automorphism $\varphi$ for $\Omega/\Sigma$, the class $\mathfrak{k}(\varphi)$ is the class of $\mathbf{p}$. This fact is established by showing that the residue of $\Phi_1$ (passage to a sufficiently large extension $\Omega_1$ of $\Omega$ with $\mathbf{P}_1 | \mathbf{p}$, and subsequent reduction of $K_1 = K\Omega_1$ modulo a prolongation of $\mathbf{P}_1$) is a power of the cited meromorphism $\Pi$ times an element of $\Sigma$. Thus, proceeding as in the classical theory, it follows that $\varphi \to \mathfrak{k}(\varphi)$ is indeed the isomorphism of the law of reciprocity for $\Omega/\Sigma$. The theory of ray classes, though technically more involved, is carried out in a similar manner. *O. F. G. Schilling.*

**Prachar, K. Verallgemeinerung eines Satzes von Hardy und Ramanujan auf algebraische Zahlkörper.** Monatsh. Math. 56, 229–232 (1952).

A well-known result of Hardy and Ramanujan is that the number of integers, $n$, less than $x$ for which

$$|v(n) - \log\log n| > (\log\log n)^{1/2+d}, \quad d > 0,$$

where $v(n)$ is the number of prime divisors of $n$, is $o(x)$ as $x \to \infty$. Using the technique of Turán [cf. Hardy and Wright, An introduction to the theory of numbers, Oxford, 1938, p. 357], the author extends this result to algebraic number fields, replacing $v(n)$ by $v(a)$, where $v(a)$ is the number of prime ideals dividing the ideal $a$, and $\log\log n$ by $\log\log N(a)$, where $N(a)$ is the norm of $a$. *R. Bellman.*

**Carlitz, L. A note on common index divisors.** Proc. Amer. Math. Soc. 3, 688–692 (1952).

Let $Z$ denote the field generated by $\zeta$, a primitive $p$th root of unity over the rational field and let $C_\nu$ be a subfield of degree $\nu$ of $Z$. The author proves some criteria for common index divisors 2 and 3 of $C_\nu$. For instance, 2 is common index divisor of $C_4$ if and only if $p \equiv 1 \pmod 8$ and 3 is common index divisor of $C_4$ if and only if $p = a^2 + b^2$, $a \equiv 1$, $b \equiv 0 \pmod 2$, $3/b$ for $p \equiv 1 \pmod 8$, $3/a$ for $p \equiv 5 \pmod 8$. Analogous criteria are given for $C_6$, $C_8$, and $C_{12}$.

*H. Bergström* (Gothenberg).

**Carlitz, L. Sums of primitive roots in a finite field.** Duke Math. J. 19, 459–469 (1952).

Let $\beta_1, \beta_2, \cdots, \beta_r$ be numbers in a finite field $GF(p^n)$ belonging to the exponents $e_i$ (where $e_i | p^n - 1$); let $\alpha, \alpha_1, \cdots, \alpha_r$

be fixed numbers in the field, with $\alpha_i \neq 0$. This paper deals in the main with the number of solutions $N_r(\alpha)$ of the equation (*) $\alpha = \alpha_1\beta_1 + \cdots + \alpha_r\beta_r$. For $r \geqq 3$, with $e_1 \leqq e_2 \leqq \cdots \leqq e_r$, it is shown that $N_r(\alpha) =$

$$\frac{\varphi(e_1)\cdots\varphi(e_r)}{p^n-1} + O[p^{n(\frac{1}{2}+\epsilon)}\varphi(e_3)\cdots\varphi(e_r)] \quad \text{as} \quad p^n \to \infty.$$

A similar result holds for $r=2$ if $\alpha \neq 0$. The proof is given first for the case $r=2$, making use of the multiplicative characters of $GF(p^n)$, and is then extended to the general case. The following generalization of (*) is also considered: $\alpha = \alpha_1\beta_1 + \cdots + \alpha_r\beta_r + \delta_1\xi_1^{k_1} + \cdots + \delta_s\xi_s^{k_s}$; here, in addition to the above mentioned conditions, $k_i \mid p^n - 1$ and $\delta_i \neq 0$. In the particular case $r=1$, $s$ even, $k_i = 2$, $p \neq 2$, an explicit formula for the number of solutions is obtained.

*H. W. Brinkmann* (Swarthmore, Pa.).

**Vandiver, H. S. On cyclotomy and extensions of Gaussian type quadratic relations involving numbers of solutions of conditional equations in finite fields.** Proc. Nat. Acad. Sci. U. S. A. 38, 981–991 (1952).

Let $p$ be an odd prime and $g$ a primitive root of a finite field of order $p^n$. Let $m_1$, $m_2$ be divisors of $p^n-1$ and write $p^n - 1 = m_1m_1' = m_2m_2'$. For fixed integers $r_1$, $r_2$ the cyclotomic number $(r_1, r_2)$ is defined as the number of solutions in the $y$'s of the relation $g^{r_1+m_1y_1} + g^{r_2+m_2y_2} = 1$ with $y_i$, $i=1, 2$, ranging over the integers $0, 1, \cdots, m_i'-1$. In the present paper the author develops a method for deriving various types of quadratic relations involving $(r_1, r_2)$. The results are extensions in a number of directions of the following formula due to Mitchell [Trans. Amer. Math. Soc. 17, 165–177 (1916)]:

$$\sum_{i=0}^{c-1}(j, i)(k-i, l-i) + A(j, k-l+\epsilon) = \sum_{i=0}^{c-1}(k, i)(j-i, l-i) + A(k, j-l+\epsilon),$$

where $c^2A(r, s)/m = \sum_{a=0}^{c-1}\alpha^{ar}\sum_{b=0}^{c-1}\alpha^{bs}$, and where $m_1 = m_2 = c$, $p^n - 1 = cm$, $\alpha = \exp(2\pi i/c)$, $\epsilon = \text{ind}(-1)$, and $j$, $k$, $l$ are arbitrary integers. The author then gives applications of his results to cyclotomy and modular relations involving binomial coefficients. Two typical theorems are as follows: (1) If $r$, $s$, $t$ are arbitrary integers, and $\psi(\alpha^r, \alpha^s) = \sum \alpha^{rh_1+sh_2}$, where $h_1$, $h_2$ extend over values in the range $1, 2, \cdots, p^n-2$ such that $g^{h_1} + g^{h_2} = 1$, then

$$\psi(\alpha^{s+r}, \alpha^t)\psi(\alpha^{-s}, \alpha^{-t}) + m\sum_{d=0}^{c-1}\alpha^{dt} = \psi(\alpha^{s+t}, \alpha^r)\psi(\alpha^{-s}, \alpha^{-r}) + m\sum_{b=0}^{c-1}\alpha^{br}.$$

(2) If $|x|$ denotes the least positive or zero residue of $x$ modulo $v$, $0 < i < v$, $i = r, s, t$, $|s+r| \neq 0$, $|s+t| \neq 0$, then the following "law of reciprocity" holds:

$$\binom{|s+r|}{v-t}\binom{v-s}{t} \equiv \binom{|s+t|}{v-r}\binom{v-s}{r} \pmod{p}.$$

*A. L. Whiteman* (Princeton, N. J.).

**Brandt, Heinrich. Binäre quadratische Formen im Gaussschen Zahlkörper.** Math. Nachr. 7, 151–158 (1952).

The author studies characters and genera of primitive quadratic forms $\varphi = \rho\xi^2 + \sigma\xi\eta + \tau\eta^2$, where $\rho$, $\sigma$, $\tau$ are Gaussian integers [cf., H. J. S. Smith, Proc. Roy. Soc. London 13, 278–298 (1864) = Collected math. papers, v. 1, Oxford, 1894, pp. 418–442]. His theory is based on the first of his two recent formulations of the quadratic reciprocity law for the Gaussian field [Comment. Math. Helv. 26, 42–54 (1952); these Rev. 13, 726], and it is much simpler than Smith's. He employs stem-discriminants (corresponding to field-discriminants for the rational base-field) and prime-discriminants ($-4$, $8$, $-8$, $\pm p$, $p$ prime, $\pm p \equiv 1 \pmod 4$, in the rational case). In the Gaussian case, however, not all stem-discriminants are products of (proper) prime-discriminants. Improper prime-discriminants, which are not discriminants, but of which an even number may occur as factors of stem-discriminants, must also be considered. An effective summary of results of Smith stated for 31 cases (loc. cit., 420–423, and Table III) is the theorem that for every proper or improper prime-discriminant which divides $\Delta = \sigma^2 - 4\rho\tau$, there is a character of $\varphi$. The proof is relatively simple. The author's necessary and sufficient condition for the existence of a genus differs more essentially from Smith's. It is as follows. Let $\Delta = \theta^2\bar{\Delta}$, where $\theta^2$ is the square discriminant factor of $\Delta$ of maximum norm. Then $\bar{\Delta}$ is a product $\bar{\Delta} = \delta_1\delta_2\cdots\delta_e$ of $e$ distinct proper or improper prime-discriminants, and if $\chi_i$ is the character associated with $\delta_i$, $i = 1, \cdots, e$, and $\mu$ is an integer represented by $\varphi$ and prime to $\Delta$, then $\chi_1(\mu)\chi_2(\mu)\cdots\chi_e(\mu) = 1$.

*R. Hull* (Lafayette, Ind.).

**Cassels, J. W. S. Über einen Perronschen Satz.** Arch. Math. 3, 10–14 (1952).

Let $f(x, y) = ax^2 + 2bxy + cy^2$, where $a$, $b$, and $c$ are complex numbers and $\Delta = b^2 - 4ac \neq 0$, and let $R(i)$ denote the ring of Gaussian integers. The author proves: There exists a universal constant $A_0 > 3/4$ such that $|f(x, y)| \leqq |\Delta/A_0|^{1/2}$ has a solution $x, y \in R(i)$, $(x, y) \neq (0, 0)$, unless $f$ is equivalent to a multiple of $x^2 + xy + y^2$. The related Perron theorem is: There exist infinitely many pairs $(x, y) \neq (0, 0)$, $x, y \in R(i)$, for which $|f(x, y)| \leqq |4\Delta/3|^{1/2}$. The author gives an independent proof of his theorem, which, however, he says can be obtained from Perron's theorem by employing the Heine-Borel theorem and the like.

*R. Hull* (Lafayette, Ind.).

**Cassels, J. W. S. The product of $n$ inhomogeneous linear forms in $n$ variables.** J. London Math. Soc. 27, 485–492 (1952).

Let $L_1, \cdots, L_n$ be $n$ real linear forms in $\xi = (x_1, \cdots, x_n)$ of determinant $\Delta \neq 0$. It has been conjectured that the minimum of $|L_1 \cdots L_n|$ for $\xi \equiv \xi_0 \pmod 1$ is always at most $2^{-n}|\Delta|$; this has been proved for $n \leqq 4$. Equality is required for the forms $L_j^* = p_jx_j$, with $p_j \neq 0$, and $\xi_0 \equiv (\frac{1}{2}, \cdots, \frac{1}{2}) \pmod 1$. The author proves that there are sets of forms $L_1^{(k)}, \cdots, L_n^{(k)}$, inequivalent to $L_1^*, \cdots, L_n^*$, for which the minimum of $|L_1^{(k)} \cdots L_n^{(k)}|$ for $\xi \equiv (\frac{1}{2}, \cdots, \frac{1}{2}) \pmod 1$ has $2^{-n}|\Delta|$ as a limit as $k \to \infty$. The result, but not the proof, extends that for $n=2$ by Davenport [Nederl. Akad. Wetensch., Proc. 49, 815–821 = Indagationes Math. 8, 518–524 (1946); these Rev. 8, 444].

*L. Tornheim.*

**Barnes, E. S. On indefinite ternary quadratic forms.** Proc. London Math. Soc. (3) 2, 218–233 (1952).

Let $Q(x, y, z)$ be a real indefinite ternary quadratic form of determinant $d \neq 0$, and let $M(Q)$ be the lower bound of $|Q(x_1, y_1, z_1) \cdot Q(x_2, y_2, z_2) \cdot Q(x_3, y_3, z_3)|$ for integral variables satisfying

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \pm 1.$$

It is proved that $M(Q) < |d|/2.2$ except when $Q$ is equivalent (under unimodular transformation) to a multiple of $Q_1$, $Q_2$, or $Q_3$ (three explicitly given forms for which $M(Q)/|d| = 2/3$,

12/25 and 12/25, respectively.) It is also shown that if $Q$ assumes values arbitrarily close to 0, then $M(Q)=0$. In proving these theorems the author uses his previous results on quadratic forms [same Proc. (3) 1, 257–283 (1951); these Rev. 13, 627], and also a lemma due to H. Davenport [Acta Math. 80, 65–95 (1948); these Rev. 10, 101].

*I. Reiner* (Urbana, Ill.).

**Lehner, Joseph. A diophantine property of the Fuchsian groups.** Pacific J. Math. 2, 327–333 (1952).

If $\alpha$ is any irrational real number and $h\leqq\frac{1}{2}\sqrt{5}$, then it is well-known that there exists an infinity of reduced fractions $p_i/q_i$ such that

$$(1)\qquad \left|\alpha-\frac{p_i}{q_i}\right|<\frac{1}{2hq_i^2}\quad (i=1, 2, 3, \cdots).$$

The author gives a proof of this for $h\leqq\frac{1}{2}\sqrt{3}$, using the theory of the modular group, and proves the following generalisation. Let $G$ be a Fuchsian group, with real axis as principal circle, which is generated by a finite number of substitutions. Let $P$, the set of parabolic points of $G$, be an infinite set including the point $\infty$. Let $\alpha\in\bar{P}-P$. Then there exists $p_i/q_i$ $(i=1, 2, \cdots)$ for which (1) is true, where $p_i/q_i\in P$ and $h=h(G)$ is a geometrical constant depending only on the group $G$. The number $h(G)$ is defined during the course of the proof, which makes use of L. R. Ford's method of constructing fundamental regions by means of isometric circles. Roughly the idea of the proof is to take a line through the point $\alpha$ perpendicular to the real axis. This passes through an infinity of fundamental regions. These are all mapped onto the principal fundamental region $R_0$ so that the line is replaced by an infinity of arcs crossing $R_0$. It is shown that an infinity of these arcs possess points which are at a distance greater than some fixed distance $h'$ from the real axis. These arcs correspond to transformation matrices with bottom row $(q_i, -p_i)$, say, and for these the inequality (1) is valid. In the case of the ordinary modular group $h=h'\geqq\frac{1}{2}\sqrt{3}$.

*R. A. Rankin* (Birmingham).

**Schmetterer, L. Notiz zu einem Satz über Diophantische Approximationen.** Monatsh. Math. 56, 253–255 (1952).

It is shown that if $\theta$ non-$\varepsilon$ $k(i)$ and $\beta$ is not real, the inequality $|y\theta-x-\beta|<|h|/2|y|$ has no solution in Gaussian integers $x, y$ with $y\neq 0$ if $0<|h|<1$. Moreover, 2 is the best possible constant, in the sense that the theorem becomes false if 2 is replaced by a larger number. This is an analogue of a theorem of Kanagasabapathy [Proc. Cambridge Philos. Soc. 48, 365–366 (1952); these Rev. 13, 825] in the real case.

*W. J. LeVeque* (Ann Arbor, Mich.).

**Apfelbeck, Alois. A contribution to Khintchine's principle of transfer.** Czechoslovak Math. J. 1(76) (1951), 119–147 (1952) = Čehoslovack. Mat. Ž. 1(76) (1951), 141–171 (1952).

The author generalises Khintchine's principle of transfer [see, for example, J. F. Koksma, Diophantische Approximationen, Springer, Berlin, 1936, Kap. V, and more recent work of Jarník, e.g., Acta Arith. 2, 1–22 (1936)]. Let $\vartheta_{ij}$ $(1\leqq i\leqq n, 1\leqq j\leqq m)$ be given real numbers and put

$$S_i(x)=\sum_{j=1}^{m}\vartheta_{ij}x_j+x_{m+i}\quad (i=1, 2, \cdots, n),$$

$$T_j(y)=\sum_{i=1}^{n}\vartheta_{ij}y_i+y_{n+j}\quad (j=1, 2, \cdots, m),$$

where the numbers $x_j$, $y_i$ are integers $(i, j=1, 2, \cdots, m+n)$. Define

$$\psi_1(t)=\min_{\substack{0<\max|x_j|\leq t\\ 1\leq j\leq m}}\left(\max_{1\leq i\leq n}|S_i(x)|\right),$$

$$\psi_2(t)=\min_{\substack{0<\max|y_i|\leq t\\ 1\leq i\leq n}}\left(\min_{1\leq j\leq m}|T_j(y)|\right),$$

$$\alpha=\sup\omega\quad\text{for which}\quad \limsup_{t\to\infty}\psi_1(t)t^{(m+\omega)/n}<+\infty,$$

$$\beta=\sup\omega'\quad\text{for which}\quad \limsup_{t\to\infty}\psi_2(t)t^{(n+\omega')/m}<+\infty.$$

The "indices" $\alpha$ and $\beta$ indicate the degree of precision to which the systems of equations $S_i(x)=0$ and $T_j(y)=0$ can be solved in integers. Khintchine and Jarník considered the case $m=1$ (or $n=1$). The following theorem is proved:

(i) $\displaystyle\beta\geqq\frac{n\alpha}{m(m+n-1)+(m-1)\alpha},\quad \alpha\geqq\frac{m\beta}{n(m+n-1)+(n-1)\beta}.$

(ii) If $m\geqq 2$ and if $\alpha>2(m+n-1)(m+n-3)$, then

$$\beta\geqq\frac{n\alpha-2n(m+n-3)}{(m-1)\alpha+m-(m-2)(m+n-3)}.$$

The extent to which these results are best possible is also investigated. The method of proof is too complicated to be adequately described here.

*R. A. Rankin.*

**Postnikov, A. G. On the distribution of the fractional parts of the exponential function.** Doklady Akad. Nauk SSSR (N.S.) 86, 473–476 (1952). (Russian)

Let $q$ be an integer greater than unity, $\alpha$ a real number, $\Delta$ a subinterval of (0, 1) of length $|\Delta|$ and $N_P(\Delta)$ the number of fractional parts $\{\alpha q^x\}$ $(x=1, 2, \cdots, P)$ which lie in $\Delta$. I. I. Šapiro-Pyateckiĭ [Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 47–52 (1951); these Rev. 13, 213] showed that $\{\alpha q^x\}$ is uniformly distributed if there exists a constant $C$ such that $\limsup_{P\to\infty}N_P(\Delta)/P<C|\Delta|$ for every $\Delta$. The author improves this result by showing that $\{\alpha q^x\}$ is uniformly distributed if, for some positive $C$ and $k$,

$$\limsup_{P\to\infty}\frac{N_P(\Delta)}{P}<C|\Delta|\left(1+\log\frac{1}{|\Delta|}\right)^k.$$

This he does by applying Vinogradov's method of estimating trigonometric sums with polynomial exponents to the sums $\sum e^{2\pi i m\alpha q^x}$. The argument is carried through in detail in an exceptionally lucid fashion.

*R. A. Rankin.*

**Gel'fond, A. O. The distribution of fractional parts and convergence of functional series with gaps.** Moskov. Gos. Univ. Učenye Zapiski 148, Matematika 4, 60–68 (1951). (Russian)

Let $L$ be an increasing sequence of real numbers $y_1, y_2, \cdots$ and $M$ a sequence of points $\tau_k=(\beta_{1k}, \beta_{2k}, \cdots, \beta_{\nu k})$ in the unit cube in $\nu$-dimensional space, so that $0\leqq\beta_{ik}<1$ for $1\leqq i\leqq\nu$, $k=1, 2, \cdots$. The fractional parts $\{f_i(y)\}$ of $\nu$ functions $f_i(y)$ are said to be $(\varphi, M)$ distributed in $\nu$-dimensional space if the set of inequalities

$$|\{f_i(y)\}-\beta_{ik}|\leqq\varphi(y)\quad (i=1, 2, \cdots, \nu)$$

[in the paper the braces { } are omitted] possess for every $\tau_k$ of $M$ an infinity of solutions in values of $y$ belonging to $L$. Here $\varphi(y)$ is any given decreasing function satisfying $0\leqq\varphi(y)\leqq 1$ for $y>0$ and $\lim_{y\to\infty}\varphi(y)=0$. Two theorems which are similar to the two theorems of an earlier paper of the author [Doklady Akad. Nauk SSSR (N.S.) 64, 437–440 (1949); these Rev. 10, 682] are proved and from them re-

sults concerning gap-series and the construction of analytic functions with prescribed fractional parts at the points $z=1, 2, 3, \cdots$ are proved. The final result concerns the Dirichlet series $f(z)=\sum_0^\infty C_n e^{-\lambda_n z}$ where $\{\lambda_n\}$ is an increasing sequence such that $\lim_{n\to\infty} n/\lambda_n=0$, and $U(z)$ is defined by $U(z)=\prod_1^\infty(1-z^2/\lambda_n^2)$. The author proves that if the series for $f(z)$ converges in the half-plane $\Re z>0$ and if $\limsup_{n\to\infty}|C_n U'(\lambda_n)|^{1/\lambda_n}=1$, then $f(z)$ cannot be continued analytically onto the imaginary axis.

The proof of the first theorem needs slight modification since, in contrast to the earlier paper, $|z_\alpha|\geqq 1$, so that the inequality (8) need not be true. *R. A. Rankin.*

**Srinivasan, M. S. Shortest semiregular continued fractions.** Proc. Indian Acad. Sci., Sect. A. **35**, 224–232 (1952).

The author points out the fact that for certain rational numbers $p/q$ there exist semiregular continued fraction expansions, other than the nearest integer continued fraction, having the minimum number of partial quotients. A recurrence formula is developed which gives the number of such shortest continued fractions in terms of the elements of the nearest integer continued fraction. It is noted that a singular continued fraction is also a shortest continued fraction, and a new proof of the existence and uniqueness of the singular continued fraction for $p/q$ is given.

*W. T. Scott* (Evanston, Ill.).

**Koksma, J. F. On continued fractions.** Simon Stevin **29** (1951/52), 96–102 (1952).

For a real number $\alpha$ let $\alpha=(b_0, b_1, \cdots)$ be a simple continued fraction expansion (not necessarily unique in case $\alpha$ is rational), and let $\{p_n/q_n\}$ be the corresponding sequence of approximants. The secondary approximants are the fractions $(bp_{n-1}+p_{n-2})/(bq_{n-1}+q_{n-2})$, $b=1, 2, \cdots, b_{n-1}$; $n\geqq 1$. The author proves that if $p/q$ $(q>1)$ satisfies $|\alpha-p/q|\leqq 1/\beta q^2$, then either (i) $p/q$ is reducible and equals an approximant of $\alpha$, or (ii) $p/q$ is irreducible and for suitable $n$ and $b$ is equal to $(bp_n+p_{n-1})/(bq_n+q_{n-1})$, $0\leqq b\leqq b_{n+1}$; moreover, $\beta\leqq 1/b+1/(b_{n+1}-b)$. This result is used to derive theorems of Legendre and Fatou. *W. T. Scott* (Evanston, Ill.).

**Bergmann, G. Periodische Ketten linearer Transformationen.** J. Reine Angew. Math. **190**, 108–124 (1952).

The author continues her earlier work on an analogue of continued fraction developments for nonsingular real 3×3 matrices **A** with real coefficients [G. Bullig, Mitt. Math. Ges. Hamburg **8**, part 2, 164–187 (1940); these Rev. **2**, 351]. The proofs depend very considerably on geometrical intuition. Only matrices satisfying a condition which corresponds to the existence of a doubly-infinite continued fraction in the two-dimensional case are considered. Two such matrices are to be called equivalent if there is an integer matrix **T** of determinant $|\mathbf{T}|=\pm 1$ with $\mathbf{A}=\mathbf{TB}$. A matrix **A** is called reduced if, in geometrical terms, the lattice generated by its three rows $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ has no points except the origin inside the rectangle with sides parallel to the coordinate planes with the origin as centre and $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ on its faces. Every matrix is equivalent to a reduced matrix. Three neighbours $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ of a reduced matrix **A** are defined which are reduced and equivalent to **A**. Similarly, $\mathbf{A}_{p_1\cdots p_n}$ is the $p_n$-neighbour of $\mathbf{A}_{p_1\cdots p_{n-1}}$. If **A**, **B** are both reduced and equivalent, then $\mathbf{B}=\mathbf{A}_{p_1\cdots p_n}$ for some $n$, $p_1\cdots p_n$. Define the integral unimodular matrix $\mathbf{T}_{p_1\cdots p_n}(\mathbf{A})=\mathbf{A}_{p_1\cdots p_n}\mathbf{A}^{-1}$. If $\mathbf{T}_{p_1\cdots p_n}(\mathbf{A})=\mathbf{T}_{p_1\cdots p_n}(\mathbf{B})$ for all $n$, $p_1, \cdots, p_n$; then $\mathbf{A}=\mathbf{B}\,\mathbf{C}$ for some diagonal matrix **C**, and **A** is said to be proportional to **B**. Finally, if a reduced matrix is equivalent to a matrix proportional but unequal to it, then it is proportional to a matrix obtained from a totally real cubic number field in a certain way. *J. W. S. Cassels* (Cambridge, England).

***Leman, A. Vom periodischen Dezimalbruch zur Zahlentheorie.** 3te Auflage, bearbeitet von B. Schoeneberg. B. G. Teubner Verlagsgesellschaft, Leipzig, 1952. 60 pp. DM 2.10.

## ANALYSIS

**Kuipers, L. Properties of some elementary functions.** Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 388–393 (1952).

Let $f(x)$ be a real differentiable function satisfying the conditions (A): $f(-a)=f(a)=0$, where $a$ is a constant; $f(x)>0$ $(-a<x<a)$; maximum $f(x)$ on $(-a, a)=M$; $y=f(x)$ $(0\leqq y<M)$ is satisfied by only two values $x_1$, $x_2$ with $-a\leqq x_1<x_2\leqq a$. The tangents of the graph of $y=f(x)$ at $x=-a$ and $x=a$ meet at a point $T$ with $y_T$ as ordinate. The "tangential triangle" is formed with $(-a, 0)$, $(a, 0)$, and $T$ as angular points. Its area is $ay_T$. Erdös and Grünwald [Ann. of Math. (2) **40**, 537–548 (1939), pp. 537–540; these Rev. **1**, 1] proved that, if $f(x)$ is a polynomial of degree $\geqq 2$ with only real roots and satisfies (A), then (B) $\int_{-a}^a f(x)dx\geqq 2ay_T/3$, and (C) $\int_{-a}^a f(x)dx\leqq 4aM/3$. Szekeres [cf. Erdös and Grünwald, loc. cit.] showed that (D) $y_T\leqq 2M$ (cf. also Kuipers [Nieuw Arch. Wiskunde (2) **23**, 243–246 (1951); these Rev. **13**, 232]). Here the theorems of Szekeres-Erdös-Grünwald are generalized as follows. (I) Let $f(x)$ be a real differentiable function with (A), and be of the form $(a^2-x^2)\phi(x)$, where $\phi(x)$ satisfies the condition

$$\phi(x_1)\phi(x_2)\leqq\phi(x_1+d)\phi(x_2-d),$$
$$-a\leqq x_1\leqq x_1+d\leqq x_2-d\leqq x_2\leqq a,$$

for each pair $x_1$, $x_2$ for which $f(x_1)=f(x_2)$, and each $d$ $(0\leqq d\leqq\frac{1}{2}(x_2-x_1))$. Then $f(x)$ satisfies (B), (C), and (D). (II) Let $f(x)$ be the function $(a^2-x^2)\phi(x)$ with $\phi(-a)$, $\phi(a)>0$ and $\phi''(x)\leqq 0$ on $-a\leqq x\leqq a$. Then $f(x)$ satisfies (B), (C), and (D). *E. Frank* (Chicago, Ill.).

**Olevskiĭ, M. N. On the Taylor-Delsarte formula and on the mean value of a function on the surface of a sphere in a space of constant curvature.** Doklady Akad. Nauk SSSR (N.S.) **86**, 657–660 (1952). (Russian)

The object of this paper is to give a broad generalization of Taylor's formula along lines similar to those used by B. M. Levitan [same Doklady (N.S.) **73**, 269–272 (1950); these Rev. **12**, 91, 1002]. By means of this generalization there is obtained a generalization of P. Pizzeti's mean-value formula [R. Courant and D. Hilbert, Methoden der mathematischen Physik, Bd. 2, Springer, Berlin, 1937, p. 260].

*H. P. Thielman* (Ames, Iowa).

**Sard, Arthur. Remainders as integrals of partial derivatives.** Proc. Amer. Math. Soc. **3**, 732–741 (1952).

The author has previously [Acta Math. **84**, 319–346 (1951); these Rev. **12**, 680] investigated the linear functionals (on functions of several variables) which occur as

remainders in approximations. In this paper he proves a new "kernel" theorem which has some advantages over the corresponding theorem of the earlier paper, though it is applicable only to certain "admissible" functionals.

*U. S. Haslam-Jones* (Oxford).

**Freud, Géza. Restglied eines Tauberschen Satzes. I.** Acta Math. Acad. Sci. Hungar. 2, 299–308 (1951). (Russian summary)

Let $v(t)$ be monotone increasing over $0\leqq t<\infty$. Let $f(t)$ be non-negative and such that $F(s)=\int_0^\infty f(t)e^{-st}dv(t)$ exists, as a Lebesgue-Stieltjes integral, for $s>0$. If $\alpha$, $c_0$, and $\epsilon$ are positive constants and $F(s)=S\Gamma(\alpha+1)s^{-\alpha}[1+r(s)]$ where $|r(s)|<c_0s^\epsilon$ when $s>0$, then there is a constant $c_1$ such that

$$\int_0^x f(t)dv(t)=Sx^\alpha[1+\beta(x)]$$

where $|\beta(x)|<c_1/\log x$ when $x>2$. The proof employs the Karamata method of polynomial approximation.

*R. P. Agnew* (Ithaca, N.Y.).

## Calculus

**Lenz, Hanfried. Zurückführung einiger Integrale auf einfachere.** S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 73–80 (1952).

Let $z_1, \cdots, z_n$ be any $n$ distinct complex numbers, and let $R(z)$ be a rational function of $z$. The author gives a table of values of $\gamma_1, \cdots, \gamma_n$ ($n=3$ or 4) for which the integral

$$\int R(z)(z-z_1)^{\gamma_1}\cdots(z-z_n)^{\gamma_n}dz$$

is reducible to elementary and elliptic integrals, or integrals of genus $\leqq 2$.

*A. Erdélyi* (Pasadena, Calif.).

**Truesdell, C. A form of Green's transformation.** Amer. J. Math. **73**, 43–47 (1951).

The transformation of a volume integral into a surface integral is familiar in 3 equivalent forms known as Gauss's, Ostrogradski's and Green's theorems. There are many other possible forms; any form, containing at least as many arbitrary functions of position as the order of the highest derivative occurring in it, is equivalent to all the others and no new results can be obtained by building up apparently more complicated combinations; but some of these, however, are useful in physics. The purpose of the present paper is to give an economical derivation of a new form sufficiently elaborate to include a rather large number of known examples. Some applications to fluid dynamics are sketched, namely the demonstration of a theorem stated, but not adequately proved, by P. Duhem [Ann. Fac. Sci. Univ. Toulouse (2) **5**, 353–404 (1903), pp. 353–376]: There can exist no continuously differentiable motion of an inviscid fluid such that: 1) the vorticity is everywhere 0 or parallel to the velocity; 2) the velocity $V$ vanishes upon all finite boundaries and in infinite regions: $r^3v^2\to 0$.

*J. Kampé de Fériet* (Lille).

***Chattelun, Lucien. Calcul vectoriel. Tome I. Algèbre. Algèbre linéaire. Applications.** Gauthier-Villars, Paris, 1952. viii+605 pp. 5000 francs.

This volume deals with the algebra of vectors and its applications to geometry and trigonometry. The content is largely elementary in nature, but the exposition assumes a moderate amount of mathematical maturity. Many topics beyond those commonly treated in "Vector Analysis" textbooks are included, and the applications to geometry are so rich that a reader could easily use the book as an introduction to "modern geometry." Although too lengthy for American courses in vectors, it is excellent background reading for teachers of such courses.

The chapter headings are: I) Vectors, definitions, linear vectorial expressions. II) Grassmann linear forms. III) Geometric applications. IV) Linear transformations and affine geometry. V) Scalar products. VI) Vector products and mixed products, composite products. Definitions and general properties. VII) Applications of vector algebra. VIII) Metric transformations and metric geometry. Contravariant and covariant components of a vector. Linear substitutions related to a change of the reference system. IX) Grassmann forms of the second kind. Alternating product. Systems of sliding vectors.

The last 200 pages consist of four notes: I) Vectors in an $n$-dimensional Euclidean space. II) Vector products in $n$-dimensional Euclidean space and determinants of order greater than 3. III) Quaternions. IV) Linear algebra. Linear operators and vector homographies.

*C. B. Allendoerfer.*

## Theory of Sets, Theory of Functions of Real Variables

**Davis, Anne C., et Sierpiński, Waclaw. Sur les types d'ordre distincts dont les carrés sont égaux.** C. R. Acad. Sci. Paris **235**, 850–852 (1952).

An example of two enumerable order types $\alpha$, $\beta$ such that $\alpha^2=\beta^2$ and $\alpha\neq\beta$ was given by Anne C. Davis [Bull. Amer. Math. Soc. **58**, 382 (1952)] and simplified by Sierpiński. If $\alpha=\omega\eta$ and $\beta=\omega(\eta+1)$, then $\alpha\neq\beta$ and $\alpha^2=\beta^2=\alpha$; in fact, $\alpha$ and $\beta$ are the only solutions of the equation $\xi^2=\alpha$, and, for every natural number $n\geqq 2$, $\alpha^n=\beta^n=\alpha$. This leads the authors to pose the following problems. (1) Are there two order types $\alpha$, $\beta$ such that $\alpha^2=\beta^2$ and $\alpha^3\neq\beta^3$? (2) Are there two order types $\alpha$, $\beta$ such that $\alpha^2\neq\beta^2$ and $\alpha^3=\beta^3$? The equation $\xi^2=(\omega^*+\omega)\eta$ has exactly three solutions: $(\omega^*+\omega)\eta$, $(\omega^*+\omega)(\eta+1)$, $(\omega^*+\omega)(1+\eta)$; and, if $\tau$ is any one of the $2^{\aleph_0}$ enumerable, scattered, unbordered order types, then the equation $\xi^2=\tau\eta$ has at least three solutions: $\tau\eta$, $\tau(\eta+1)$, $\tau(1+\eta)$. Tarski has communicated to the authors that under the continuum hypothesis there exists a nonenumerable order type $\alpha$, viz., $\omega_1\eta_1$ (where $\eta_1$ is a Hausdorff normal type) for which the equation $\xi^2=\alpha$ admits of two solutions $(\omega_1\eta_1, \omega_1(\eta_1+1))$.

*F. Bagemihl* (Rochester, N. Y.).

**Davis, Anne C. Sur l'équation $\xi^n=\alpha$ pour des types d'ordre.** C. R. Acad. Sci. Paris **235**, 924–926 (1952).

For every cardinal number $m=0, 1, 2, \cdots, \aleph_0, 2^{\aleph_0}$, there exist $2^{\aleph_0}$ enumerable order types $\alpha$ such that, if $n$ is an arbitrary natural number not less than 2, the equation $\xi^n=\alpha$ has exactly $m$ solutions.

*F. Bagemihl.*

**Gottschalk, W. H. The extremum law.** Proc. Amer. Math. Soc. **3**, 631 (1952).

The author states five new forms of the extremum law (or maximum principle), by which is meant the class of statements equivalent to the axiom of choice and asserting the existence of a maximal or minimal element. For example, Zorn's lemma and the maximal chain theorem are cases of

the extremum law. The new forms were suggested by the formulation of A. D. Wallace [Bull. Amer. Math. Soc. **50**, 278 (1944); these Rev. **5**, 198], which asserts the existence of a maximal asymmetric subrelation for each binary relation.

Three of the author's new forms state the existence of maximal subrelations of a given binary relation which are coherent, chained, and antisymmetric, respectively. (These terms are defined.) The other two assert the existence of a maximal set $A$ whose square $A \times A$ is contained in a given binary relation and of a maximal set whose $n$th power is contained in a given $n$-ary relation for $n$ an integer greater than one. All these forms may be modified by requiring the maximal relation to contain a suitable given relation. The author believes that the equivalence of these forms with each other and with familiar forms of the maximum principle is clear. *O. Frink* (State College, Pa.).

**Vaughan, Herbert E. Well-ordered subsets and maximal members of ordered sets.** Pacific J. Math. **2**, 407–412 (1952).

Kuratowski [Fund. Math. **3**, 76–108 (1922)] and Milgram [Rep. Math. Colloquium (2) **1**, 18–30 (1939)] have given procedures for avoiding the use of transfinite numbers in mathematical proofs. The most important theorem serving the same purpose is Zorn's lemma. Recently several proofs have been published of Zorn's lemma as a consequence of the axiom of choice. One of these proofs, patterned on Zermelo's first proof of the well-ordering theorem, was given by the reviewer [Publ. Math. Debrecen **1**, 254–257 (1950); these Rev. **12**, 485]. Another proof follows from a theorem of Bourbaki [Arch. Math. **2**, 434–437 (1951); these Rev. **13**, 923]. In the present paper, by a further exploitation of the reviewer's method, the author proves a theorem of wide range containing a common generalization of the mentioned results of Kuratowski, Milgram, and Bourbaki. Moreover, many corollaries, in particular, a new analogue of Zorn's lemma, are deduced from the main result.

*T. Szele* (Debrecen).

**Bledsoe, Woodrow W., and Morse, A. P. Some aspects of covering theory.** Proc. Amer. Math. Soc. **3**, 804–812 (1952).

The paper is mainly concerned with the correction of some errors which occurred in an earlier paper by A. P. Morse [Trans. Amer. Math. Soc. **61**, 418–442 (1947); these Rev. **8**, 571]. The most important of these was an assertion that, if $\mathfrak{K}$ is a countable family of sets in a metric space, if $\phi$ is a measure, and if a family $\mathfrak{F}$ of sets has the $\phi$ Vitali property with respect to each member of $\mathfrak{K}$, then $\mathfrak{F}$ has the $\phi$ Vitali property with respect to the union of all the sets in $\mathfrak{K}$. The authors show by an example that this assertion is false (even when $\mathfrak{K}$ contains only two sets), but that it may be corrected by adding the further condition that each member of $\mathfrak{K}$ is of finite $\phi$ measure.

*U. S. Haslam-Jones* (Oxford).

**Cafiero, Federico. Sulle famiglie di funzioni additive d'insieme, uniformemente continue.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 155–162 (1952).

Let $\Phi$ be a family of set functions $\varphi(M)$ which are totally additive in the $\sigma$-field $\mathfrak{F}$. Then $\Phi$ is called equi-continuous (according to R. Caccioppoli) if to every given sequence $\{M_n\}$ of sets of $\mathfrak{F}$ with empty limit and to every $\epsilon > 0$ there exists an index $n_0$ such that $|\varphi(M_n)| < \epsilon$ for $n \geqq n_0$ and for all $\varphi$ of $\Phi$. The author proves the following two theorems concerning sequences $\{\varphi_n(M)\}$ of totally additive set functions in $\mathfrak{F}$ (the second one is a consequence of the first one): (I) In order that the sequence $\{\varphi_n(M)\}$ be equi-continuous, it is (necessary and) sufficient that to every sequence $\{M_r\}$ of disjoint sets of $\mathfrak{F}$ and to every $\epsilon > 0$ one can find a set $M_{r_0}$ (of this sequence) and an index $n_0$ such that $|\varphi_n(M_{r_0})| < \epsilon$ for $n \geqq n_0$. (II) Let the sequence $\{\varphi_n(M)\}$ be given. If in every sequence $\{M_r\}$ of disjoint sets of $\mathfrak{F}$ there is a set on which $\{\varphi_n\}$ converges, then the sequence $\{\varphi_n(M)\}$ is equi-continuous. From (II) there results the well-known theorem [O. Nikodým, C. R. Acad. Sci. Paris **192**, 727 (1931); Monatsh. Math. Phys. **40**, 418–426, 427–432 (1933)]: If $\{\varphi_n\}$ is a sequence of totally additive set functions in $\mathfrak{F}$ and if for every $M \in \mathfrak{F}$ there exists a finite limit $\varphi(M) = \lim_{n\to\infty} \varphi_n(M)$, then $\varphi$ is also totally additive in $\mathfrak{F}$. *A. Rosenthal.*

**Hewitt, Edwin. Integration on locally compact spaces. I.** Univ. Washington Publ. Math. **3**, 71–75 (1952).

Let $C_+$ be the set of all nonnegative continuous functions defined on a locally compact Hausdorff space $X$ and vanishing outside some closed compact set. The author derives a representation of an arbitrary nonnegative linear functional $M$ on $C_+$ as the integral over $X$ with respect to its Radon measure. He is unaware of thus duplicating the work of I. E. Segal [J. Indian Math. Soc. (N.S.) **13**, 105–130 (1949); these Rev. **11**, 425]. He also discusses the relation of the resulting extension of $M$ to its slight modification described in the book of Halmos [Measure theory, Van Nostrand, New York, 1950; these Rev. **11**, 504]. *H. M. Schaerf.*

**Hewitt, Edwin, and Zuckerman, H. S. Integration in locally compact spaces. II.** Nagoya Math. J. **3**, 7–22 (1951).

The extension of the functional $M$ constructed in part I is shown to coincide with another extension obtained by H. Cartan [Bull. Soc. Math. France **69**, 71–96 (1941); these Rev. **7**, 447] on the set of all nonnegative functions for which either one of these extensions is finite.

*H. M. Schaerf* (St. Louis, Mo.).

**Areškin, G. Ya. On passage to the limit under the Lebesgue-Radon integral sign.** Soobščeniya Akad. Nauk Gruzin. SSR. **10**, 69–76 (1949). (Russian)

Let $L$ be a Borel field of subsets, including the whole space, let $f_k$ be $L$-measurable functions, and let $\mu_k$ be measures on $L$ converging to a measure $\mu$. The $f_k$ are said to have uniformly absolutely continuous integrals with respect to the $\mu_k$ if given $\epsilon > 0$ there is a $\delta > 0$ such that for each $k$ and each $H \in L$ with $\mu_k(H) < \delta \int_H f_k d\mu_k < \epsilon$. Theorem 2 asserts that if, in addition, $f_k$ converges ae$(\mu)$ to $f$, then $f$ is $\mu$-summable and $\int_H f d\mu = \lim_k \int_H f_k d\mu_k$ for each $H \in L$. Theorem 1 has the same conclusion, assuming that the $f_k$ are uniformly bounded and converge in measure-$\mu_k$ to the $L$-measurable function $f$, that is, that for each $\epsilon > 0$ $\lim_k \mu_k\{x \mid |f(x) - f_k(x)| \geqq \epsilon\} = 0$. *M. M. Day.*

**Gordon, W. L., and McArthur, C. W. A theorem on uniform Cauchy points.** Amer. J. Math. **74**, 764–768 (1952).

The authors strengthen some results of Pettis [same J. **73**, 602–614 (1951); these Rev. **13**, 217]. Using Pettis's terminology (op. cit., given also in cited review) their principal result is that, if each $f_\lambda$ is continuous, and if $\phi \geqq \max(|A|, |\Lambda|)$, then $C - C_u$ is an $I_\phi$-set. Pettis's main results are immediate corollaries of this. Applications are also considered. *A. F. Ruston* (London).

**Helsel, R. G., and Levine, N. Absolutely continuous product transformations of the plane.** Duke Math. J. 19, 595–603 (1952).

Given bounded continuous plane transformations $T_i$ defined on bounded domains $\mathfrak{D}_i$ in the $w_i$-plane, $i=1, 2$, such that $T_1\mathfrak{D}_1$ is a subset of $\mathfrak{D}_2$. Then $T_2T_1$ is a bounded continuous transformation on $\mathfrak{D}_1$. Let $\mathfrak{B}_i$ be base sets in $\mathfrak{D}_i$ such that $T_1\mathfrak{B}_1$ is a subset of $\mathfrak{B}_2$. (For definitions see T. Radó, Length and area [Amer. Math. Soc. Colloq. Publ. vol. 30, New York, 1948; these Rev. 9, 505].) If the transformations $T_i$ are absolutely continuous in $\mathfrak{D}_i$ with respect to the base sets $\mathfrak{B}_i$—briefly, AC $\mathfrak{B}_i$ in $\mathfrak{D}_i$—then they possess summable derivatives $D(w_i, T_i, \mathfrak{B}_i)$ on $\mathfrak{D}_i$, $i=1, 2$. Assuming that both of the $T_i$ are AC $\mathfrak{B}_i$ in $\mathfrak{D}_i$, it is shown that $T_2T_1$ is AC $\mathfrak{B}_1$ in $\mathfrak{D}_1$ if and only if $D(w_1, T_1, \mathfrak{B}_2)\, D(T_1w_1, T_2, \mathfrak{B}_2)$ is summable on $\mathfrak{D}_1$; moreover, if $T_2T_1$ is AC $\mathfrak{B}_1$ in $\mathfrak{D}_1$, then the chain rule for differentiation

$$D(w_1, T_2T_1, \mathfrak{B}_1)=D(w_1, T_1, \mathfrak{B}_1)\, D(T_1w_1, T_2, \mathfrak{B}_2)$$

holds almost everywhere in $\mathfrak{D}_1$. A number of special results are derived using this general theorem. In particular, if one chooses the base sets $\mathfrak{B}_i$ to be the domains $\mathfrak{D}_i$, one obtains a result for transformations absolutely continuous in the sense of Banach. *P. V. Reichelderfer* (Columbus, Ohio).

**Reifenberg, E. R. Parametric surfaces. I. Area.** Proc. Cambridge Philos. Soc. 47, 687–698 (1951).

Let $H$ be the 2-dimensional spherical Hausdorff measure over Euclidean 3-space $E_3$. For any function $f$ into $E_3$ and for any point $y$ of $E_3$ let $N(f, y)$ be the number (possibly $\infty$) of elements $x$ of the domain of $f$ for which $f(x)=y$. Furthermore, let $W(f)=\int N(f, y)dHy$.

Consider a (continuous) mapping $\varphi$ of a (compact) 2-cell $X$ into $E_3$. For each 2-cell $D$ contained in $X$ let $\mu(\varphi, D)$ be the infimum of $W(f)$ for all those mappings $f$ of $D$ into $E_3$ which agree with $\varphi$ on the boundary of $D$. Then let $A(\varphi)$ be the supremum of $\sum_{i=1}^{n}\mu(\varphi, D_i)$, where $D_1, D_2, \cdots, D_n$ are disjoint 2-cells contained in $X$. Also let $L(\varphi)$ be the Lebesgue area of $\varphi$.

Assuming that $\varphi$ is a light mapping, the author claims to prove the following three principal theorems: (1) $A(\varphi)\leqq L(\varphi)$. (2) If $L(\varphi)<\infty$, then $A(\varphi)=L(\varphi)$. (3) $W(\varphi)\geqq L(\varphi)$.

These assertions are correct. The inequality (3) has been established by the reviewer [Proc. Nat. Acad. Sci. U. S. A. 37, 90–94 (1951); these Rev. 13, 831]. Furthermore it can be shown that $A(\varphi)=L(\varphi)$ in all cases, a result much stronger than (1) and (2).

The reviewer knows no proofs of the statements (1), (2), and (3) which do not make use of practically the whole theory of area developed during the last twenty-five years by T. Radó, C. B. Morrey, L. Cesari, and the reviewer. The paper under review attempts to avoid most of this existing theory, but unfortunately contains serious gaps. For instance: (i) Use is made of A. S. Besicovitch's treatment of the problem of Plateau [Proc. Cambridge Philos. Soc. 44, 313–334 (1948); 45, 5–13, 14–23 (1949); J. London Math. Soc. 23, 241–246 (1948); these Rev. 10, 520, 521], which contains gaps never explained by that author. (ii) Use is made of various additivity properties of the Lebesgue area, such as additivity with respect to cutting of a surface by a curve of finite length, and the inequality

$$L(\varphi|P)\leqq\sum_{i=1}^{n}L(\varphi|Q_i),$$

where $P, Q_1, Q_2, \cdots, Q_n$ are finitely connected regions bounded by Jordan curves and $P\subset\bigcup_{i=1}^{n}Q_i\subset X$. (The fact that the function $L^*$ defined on page 688 is a Carathéodory outer measure depends on the last inequality.) With regard to these highly non-trivial properties of the Lebesgue area the paper under review contains no explanation whatsoever. The gaps can be filled, but in the reviewer's opinion the difficulties encountered in doing so lie much deeper than the problems actually solved by the author.

*H. Federer* (Providence, R. I.).

**Reifenberg, E. R. Parametric surfaces. II. Tangential properties.** Proc. Cambridge Philos. Soc. 48, 46–69 (1952).

Assuming as in the paper reviewed above that $\varphi$ is a light mapping of a 2-cell $X$ into $E_3$, the author considers a point $x$ of $X$ and a plane $\pi$ through $\varphi(x)$. He gives several alternate definitions, each describing reasonable conditions under which the plane $\pi$ shall be called an approximate tangent plane of $\varphi$ at $x$.

The principal theorems of the paper state that if the Lebesgue area of $\varphi$ is finite and if $T$ is the set of all those points of $X$ at which $\varphi$ has an approximate tangent plane (according to any one of the alternate definitions), then $L(\varphi)=\int N(\varphi|T, y)dHy$.

Unfortunately the reasoning offered in this paper is open to the same objections which were raised in the preceding review. Furthermore, the reviewer is unable to follow the proof of Lemma 3, from which the author derives the existence of approximate tangent planes. On the other hand, the reviewer has no doubt that the principal theorems are correct; in fact he has established similar results by different methods, applying also to the case in which $L(\varphi)=\infty$ [Bull. Amer. Math. Soc. 57, 272, 273 (1951); 58, 306–378 (1952); these Rev. 14, 149]. *H. Federer.*

## *Theory of Functions of Complex Variables*

**Glover, I. E. Analytic functions with an irregular linearly measurable set of singular points.** Canadian J. Math. 4, 424–435 (1952).

This paper contains an attempt to construct an analytic function having an assigned irregular linearly measurable point-set as the set of its singularities. However, the obscurity of the language, the number of misconceptions, and the incorrectness of the results render a detailed review unnecessary. *H. G. Eggleston* (Swansea).

**Wilson, R. A note on a theorem of Pólya's.** Quart. J. Math., Oxford Ser. (2) 3, 145–150 (1952).

The theorem of Pólya is the following: Let $f(z)$ be an analytic function and let $\lim_{n\to\infty}|b_n|^{1/n}=0$. Then $f^*(z)=\sum_0^\infty(-1)^\nu b_\nu f^{(\nu)}(z)/\nu!$ is a regular analytic function in the entire region of existence of $f(z)$ and every "zugängliche" singularity of $f(z)$ is also singular for $f^*(z)$. A zugängliche singularity is one that is accessible, in a sense difficult to describe in few words. The author defines a unique singularity to be a singularity which lies on the circle of convergence when $f(z)$ is expanded in powers of $z$ (or of $z^{-1}$) and which is interior to an arc of that circle having no other singularities. He then proves the following: Let $f(z)$ have a finite nonzero radius of convergence. Then every unique singularity of $f(z)$ is a unique singularity of $f^*(z)$, and $f^*(z)$ is regular at every regular point of $f(z)$ on its circle of convergence. The proof is made to depend on the following

theorem, also established: The product of two integral functions of order 1, one of mean type and the other of minimum type, has the same unique directions of strongest growth as the former. Extensions are given, and the connection with known results is carefully indicated. *R. M. Redheffer.*

**Bourion, Georges. Sur le prolongement analytique des séries lacunaires.** J. Math. Pures Appl. (9) **31**, 127–140 (1952).

Using the methods of LeRoy and Lindelöf the author gives examples of power series with an infinite number of gaps which can be continued analytically into certain neighborhoods of the point at infinity. By the use of theorems of Cartwright on the zeros of entire functions, results are obtained for lacunary series. The gaps in the series are caused by the vanishing of the coefficients of index $n$ where $m_k \leqq n \leqq n_{k+1}$, $k=1, 2, \cdots$, and lim inf $n_{k+1}/m_k > 1$. Domains of overconvergence are also considered. *N. Levinson.*

**Tsuji, Masatsugu. On the remainder term of Nevanlinna's second fundamental theorem.** J. Math. Soc. Japan **4**, 31–36 (1952).

Integral inequalities for the remainder in Nevanlinna's second fundamental theorem [Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 240] are obtained by the use of devices due to Ahlfors [Soc. Sci. Fenn. Comment. Phys.-Math. **8**, no. 10 (1935)]. *A. J. Macintyre.*

**Komatu, Yûsaku. The order of the derivative of a meromorphic function.** Proc. Japan Acad. **27**, 317–320 (1951).

A new proof is given of Whittaker's theorem that any meromorphic function is of the same order as its derivative. *W. Seidel* (Princeton, N. J.).

**Finn, Robert. Sur quelques généralisations du théorème de Picard.** C. R. Acad. Sci. Paris **235**, 596–598 (1952).

Let $w(z)=w(x+iy)=u(x, y)+iv(x, y)$ be of class $C'$ and let $Q(x, y)$ denote the ratio of $\limsup_{\Delta z\to 0} |\Delta w/\Delta z|$ to $\liminf_{\Delta z\to 0} |\Delta w/\Delta z|$. Then $e=Q+Q^{-1}$ is called the "eccentricity" of the mapping $w(z)$. A function $f(z)$ is said to be of class $[e]$ if its eccentricity is $\leqq e$. The author defines $E$, $F$, $G$, $J$, by $J=u_xv_y-u_yv_x$, $EJ=u_x^2+v_x^2$, $-FJ=u_xu_y+v_xv_y$, $GJ=u_y^2+v_y^2$. If $u+iv$ is defined and non-constant on the $z$-plane, and if $u+iv$ takes on neither the value 0 nor 1, then the modular function $U+iV$ of $u+iv$ is a bounded non-constant function defined everywhere. Moreover, $U+iV$ satisfies $V_y=GU_x+FU_y$, $-V_x=FU_x+EU_y$, $EG-F^2=1$, $e=E+G$, $E^2+2F^2+G^2=e^2-2$. If $D_r$ is a disc bounded by the circle $C_r$, of radius $r$, then one finds

$$H(r)=\iint_{D_r} (GU_x^2+2FU_xU_y+EU_y^2)dxdy=\oint_{C_r} UdV,$$

and hence

$$H^2(r)<\frac{dH}{dr}\int_{C_r} U^2erd\theta.$$

An integration then shows $\int_1^\infty[r \operatorname{Max}_\theta e]^{-1}dr$ converges, and hence $e$ is unbounded.

More generally, let $w(z)$ be of class $[e]$ in $D_r$. If $\alpha(\rho)$ is the oscillation of $u(x, y)$ in $D_\rho$, $\rho<R$, and if

$$\mathfrak{D}[r]=\iint_{D_r} (U_x^2+U_y^2)dxdy,$$

then

$$\alpha^2(\rho)<2\pi\mathfrak{D}[R_1]\cdot\left[\log\frac{R_1}{\rho}\right]^{-1}, \quad \mathfrak{D}[R_1]<K\left[\log\frac{R}{R_1}\right]^{-1},$$

where $K=K[\sup (u^2e) \text{ in } D_R]$. Hence every bounded collection of class $[e]$ is normal. One can now show by Montel's method that a function of class $[e]$ cannot be bounded near an isolated singularity.

The author applies the idea of eccentricity to differential equations of the form $\alpha\phi_{xx}+2\beta\phi_{xy}+\gamma\phi_{yy}=0$, where $\alpha$, $\beta$, $\gamma$ are functions of $x$, $y$, $\phi$, $\phi_x$, $\phi_y$, where $\alpha>0$, $\alpha\gamma-\beta^2>0$. He obtains the following result. Let $e$ be the eccentricity of $\phi_x-i\phi_y$ and suppose $e<K(1+\phi_x^2+\phi_y^2)^n \log(1+x^2+y^2)$, where $K\leqq 1$, $n>1$. If $\phi(x, y)$ is a solution of the d.e., regular in the whole plane, and if $\phi_x-i\phi_y$ omits a set of values of positive capacity, then $\phi(x, y)$ is a linear function. Several other applications of the notion of eccentricity are indicated. *M. O. Reade* (Ann Arbor, Mich.).

**Caccioppoli, Renato. Sur une généralisation des fonctions analytiques et des familles normales.** C. R. Acad. Sci. Paris **235**, 116–118 (1952).

**Caccioppoli, Renato. Sur une généralisation des fonctions analytiques.** C. R. Acad. Sci. Paris **235**, 228–229 (1952).

Let $w(z)=u(x, y)+iv(x, y)$ be continuous for $z=x+iy$ in a domain $D$, let $u(x, y)$ and $v(x, y)$ be absolutely continuous in $x(y)$ for almost all $y(x)$, and let the partial derivatives of $u(x, y)$ and $v(x, y)$ have summable squares. Let

$$\bar{\Phi}(x, y)=\overline{\lim_{\Delta z\to 0}}\left|\frac{\Delta w}{\Delta z}\right|.$$

Then $w(z)$ is said to be pseudo-analytic of class $c_\mu$, $\mu\leqq 1$, if

(i) $$J\left(\frac{u, v}{x, y}\right)>0, \quad \underline{\Phi}(x, y)\geqq\mu\bar{\Phi}(x, y)$$

hold almost everywhere in $D$.

Let $\Delta$ denote a domain (with area $\|\Delta\|$) with rectifiable boundary $\Gamma$ (of length $\|\Gamma\|$) contained in $w(D)$, and let $w(z)$ satisfy the additional hypothesis that there exists a constant $K$ such that

(ii) $$\|\Delta\|\leqq K\|\Gamma\|.$$

The author then obtains the following key result for pseudo-analytic $w(z)$ satisfying (ii). If $C(r)$ is a disc of radius $r$ in $D$, if $F(r)$ is the area of the image $w(C(r))$, and if $\omega(r)$ is the oscillation of $w(z)$ in $C(r)$, then

(iii) $$F(r)<\frac{2\pi K^2}{\mu}\cdot\frac{1}{\log R/r},$$

$$\omega^2(\rho)<\frac{\pi}{2\mu}F(r)\frac{1}{\log r/\rho}, \quad R>r>\rho.$$

Therefore, (iii) yields a modulus of uniform continuity for $w(z)$ in class $c_\mu$, and since the uniform limit of a sequence of functions of class $c_\mu$ is again in $c_\mu$, it follows that (ii) is a sufficient condition for a family of functions of class $c_\mu$ to be normal (in the sense of Montel). He then extends the Bloch criterion for normal families to pseudo-analytic functions as well as extending Bloch's criterion concerning exceptional values. He remarks that one may now extend Picard's theorem to pseudo-analytic functions by simply using Montel's classic argument.

The author sets $r=(R\rho)^{1/2}$ in (iii), to obtain

$$\text{(iv)}\qquad \omega(\rho)=\frac{2\pi K}{\mu}\cdot\frac{1}{\log (R/\rho)}.$$

If $w(z)$ is a function obtained by means of a Bloch transformation applied to a pseudo-analytic function omitting values 0, 1, then (iv) is satisfied. Hence he has a generalization of Schottky's theorem.

He next considers the problem of minimizing $F(r)$ to obtain

$$\text{(v)}\qquad r^{-2\mu}F(r)\geqq r_0^{-2\mu}F(r_0),\quad r>r_0.$$

He calls $H(P, r_0)\equiv r^{-2\mu}F(r)$ a Hölder "coefficient of dilation" with exponent $\mu$ for $C(r)$. From (iii) and (v) he obtains

$$\text{(vi)}\qquad H(P, r_0)>\frac{2\pi K^2}{\mu}\left(r^{2\mu}\log\frac{R}{r}\right)^{-1},\quad R>r>r_0.$$

If $2\pi K^2\mu^{-1}\lambda(R)$ denotes the minimum of the right-hand member of (vi), then $K^2<\mu(2\pi\lambda(R))^{-1}H(P, r_0)$, and hence there is an inferior limit for the maximum radius $\delta$ of the circles in $w(C(R))$. Moreover, almost everywhere, $\lim_{r\to 0} H(P, r)=0$, for $\mu<1$, and $\lim_{r\to 0} H(P, r)=\pi|w'(P)|^2$ for $\mu=1$; this extends Bloch's theorem with $H$ in place of the square of the absolute value of the derivative of the mapping function. The author also obtains a similar generalization of a theorem due to Landau. *M. O. Reade.*

**Yosida, Tokunosuke. On the behaviour of a pseudo-regular function in a neighbourhood of a closed set of capacity zero.** Proc. Japan Acad. **26**, no. 10, 1–8 (1950).

The author extends to pseudo-regular functions (the mappings defined by such functions are sometimes also called quasi-conformal; for terminology, see O. Teichmüller [Deutsche Math. **3**, 621–678 (1938)]) a number of theorems which deal with the behavior of analytic functions in the neighborhood of sets of zero capacity. Let $E$ be a bounded closed set of zero capacity in the complex $z$-plane. It is known [G. C. Evans, Monatsh. Math. Phys. **43**, 419–424 (1936), where the result is obtained in three dimensions] that there exists a positive mass distribution $\mu(e)$ on $E$ of total mass unity such that the corresponding logarithmic potential $u(z)=-\int_E \log|z-a|d\mu(a)$ becomes positively infinite at every point of $E$. Let $C_u$ denote the level curve, or curves, $u(z)=u$, where $u$ is a constant. As an example of the type of result obtained by the author, we mention the following: Let $R$ be a finite region in the $z$-plane bounded by a Jordan curve $C$ and a closed set $E$ of zero capacity lying wholly interior to $C$ and let $\zeta=f(z)$ be a function which maps $R$ in a one-to-one and pseudo-conformal manner on a finite region $G$ in the $\zeta$-plane bounded by a Jordan curve $L$ and a closed set $F$ of positive capacity lying wholly interior to $L$, so that $C$ corresponds to $L$ in this mapping. Then the integral $\int^\infty(1/D(u))du$ converges, where $D(u)$ denotes the maximum of the "dilatation quotient" of $f(z)$ on the level curve $C_u$ of Evans' potential. This is related to earlier work of Teichmüller's. The author also obtains the following extension of the maximum-modulus principle: Let $f(z)$ be bounded and pseudo-regular in a simply connected region $R$. If for every boundary point $z_0$, except perhaps for a set of zero capacity, $\limsup_{z\to z_0,\, z\in R}|f(z)|\leqq M$, $M$ a positive constant, and if $\int^\infty(1/D(u))du$ diverges, where $D(u)$ denotes the least upper bound of the "dilatation quotient" of $f(z)$ on that part of the level curve $C_u$ of Evans' potential which lies in $R$, then $|f(z)|\leqq M$ in $R$. [This generalizes a result of R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, pp. 134–135.] Other results of this nature are obtained. *W. Seidel* (Princeton, N. J.).

**Yosida, Tokunosuke. Theorems on the cluster sets of pseudo-analytic functions.** Proc. Japan Acad. **27**, 268–274 (1951).

Let $R$ be a region in the $z$-plane with boundary $B$, $E$ a bounded closed set of points of zero capacity on $B$, and $z_0$ a point of $E$. Let $w=f(z)$ be a single-valued function pseudo-analytic in $R$. If $C(f, z_0)$ denotes the cluster set with respect to $R$ of $f(z)$ at $z_0$, the author introduces the set

$$C_{B-E}(f, z_0)=\bigcap_r\overline{\bigcup_{z'}}\{C(f, z'),\ z'\in B-E,\ |z'-z_0|<r\},$$

where $\overline{\bigcup}$ is the closure of the union. Next, the author considers the level curves $L_r$: $u(z)=\log r$, $0<r<+\infty$, of Evans' potential $u(z)$ [see the preceding review]. A single-valued, pseudo-analytic function $w=f(z)$ in $R$ is said to be of class (P) if the integral $\int^\infty(1/rD(r))dr$ diverges, where $D(r)$ denotes the least upper bound of the dilatation quotient [see the preceding review] of $f(z)$ on that part of $L_r$ which lies in $R$. Two theorems are proved. Let $f(z)$ belong to (P). I) The set $\Omega=C(f, z_0)-C_{B-E}(f, z_0)$ is open. If $\Omega$ is non-empty, then $f(z)$ assumes every value in $\Omega$, except perhaps for a set of zero capacity, infinitely often in every neighborhood of $z_0$. II) If the set $E$ is contained in a finite number of components of $B$ and $\Omega$ is non-empty, then $f(z)$ assumes every value belonging to any component of $\Omega$, with at most two exceptional values, infinitely often in any neighborhood of $z_0$. *W. Seidel* (Princeton, N. J.).

**Koseki, Ken'iti. Über die Ausnahmewerte der meromorphen Funktionen.** Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 7–40 (1952).

The paper discusses the order and the distribution of values of a function meromorphic near an angular point of opening $\alpha$ on the boundary of its region of existence. The results are based on properties of conformal representation (proved in the text), the known theory of functions meromorphic in an angle [e.g., G. Valiron, J. Math. Pures Appl. (9) **10**, 457–480 (1931)] and on certain direct arguments. The following theorem generalising one of Valiron [loc. cit.] is reached. The exponent of convergence of the zeros of $f(z)-a$ has the same value $\rho$ for all $a$ with at most two exceptions. If $\lambda$ is the order of $f(z)$, then $\lambda\leqq\rho\leqq\lambda+(\pi/\alpha)$. The precise definitions of order and exponent of convergence are too long to be reproduced. *A. J. Macintyre.*

**Shah, S. M. Exceptional values of entire and meromorphic functions.** Duke Math. J. **19**, 585–593 (1952).

This is a continuation of the author's work on what he calles $E$-exceptional values [for the definition see Compositio Math. **9**, 227–238 (1951); these Rev. **13**, 452]. An $E$-exceptional value for an entire function of finite order can occur only when the order $\rho$ is an integer, and then a necessary condition is that (i) $\rho_1(\alpha)<\rho=q(\alpha)$ or (ii) $\rho_1(\alpha)=\rho=q(\alpha)$, $p(\alpha)=\rho-1$. Here $\rho_1(\alpha)$, $q(\alpha)$, $p(\alpha)$ are the exponent of convergence of the $\alpha$-points, the order of the exponential part, and the genus of the canonical product in the Hadamard factorization of $f(z)-\alpha$. The author showed previously that (i) is sufficient; here he shows that (ii) is not sufficient. He also proves additional theorems, including the following. If $\alpha$ is an $E$-exceptional value for an entire function of order $\rho$, then $f(z)$ has $\rho$ finite asymptotic values (all equal to $\alpha$); but a meromorphic function of order $\rho$ can have $E$-exceptional values which are not asymptotic. A

Borel exceptional value is an $E$-exceptional value for entire, but not for meromorphic, functions; however, if a meromorphic function has two Borel exceptional values, both are $E$-exceptional. If an entire function $f(z)$ of finite order has an $E$-exceptional value $\alpha$, then $n(r, \beta)r^{-\rho}$ approaches a limit for every $\beta \neq \alpha$, and $T(r, f) \rightarrow T(r, f')$.

*R. P. Boas, Jr.* (Evanston, Ill.).

**Shah, S. M. A note on entire functions of perfectly regular growth.** Math. Z. 56, 254–257 (1952).

Let $f(z)$ be an entire function of order $\rho$. With $L(r)$ "slowly changing" (continuous and $L(cr) \sim L(r)$) the author introduces the notations $T$ and $t$ for the lim sup and lim inf of $r^{-\rho}\{\log M(r)\}/L(r)$, and $\gamma$ and $\delta$ for the corresponding things with $\log M(r)$ replaced by $\nu(r)$ (the rank of the maximum term). He shows: (i) if $0<t\leqq T<\infty$, then $0<\delta\leqq\gamma<\infty$, and conversely; (ii) in this case there is a number $K(t, T)$ such that the lim sup and lim inf of $\{\log M(r)\}/\nu(r)$ lie between $1/(\rho k)$ and $k/\rho$; (iii) if $\log M(r) \sim Tr^\rho L(r)$ $(0<T<\infty)$, then $\nu(r) \sim T r^\rho L(r)$, and conversely. He discusses the possible cases when one of the quantities $T$, $t$, $\gamma$, $\delta$ is 0 or $\infty$. Finally he shows that his results imply that if $\rho<\infty$ and $M^{(n)}(r)$ is the maximum modulus of $f^{(n)}(z)$, then

$$\limsup \frac{\log M^{(n)}(r)}{\nu(r) \log r} = \limsup \frac{\log M(r)}{\nu(r)\log r} \leqq 1.$$

*R. P. Boas, Jr.* (Evanston, Ill.).

**Tammi, Olli. On the maximalization of the coefficients o schlicht and related functions.** Ann. Acad. Sci. Fennicae Ser. A. I. Math.-Phys. no. 114, 51 pp. (1952).

Let

$$(1) \qquad f(z)=z+\sum_{n=2}^{\infty} a_n z^n, \quad \frac{zf'(z)}{f(z)}=1+\sum_{n=1}^{\infty} b_n z^n,$$

where $f(z)$ is regular in $|z|<1$, $f(z)\neq 0$ for $0<|z|<1$. It is shown that $|b_n|\leqq \rho_\nu/\pi$ $(\nu=1, 2;\ n=1, 2, \cdots)$ where

$$\rho_1=\lim_{r\rightarrow 1}\int_0^{2\pi}|d \arg f(re^{i\varphi})|, \quad \rho_2=\lim_{r\rightarrow 1}\int_0^{2\pi}|d\log|f(re^{i\varphi})||.$$

In the case $\nu=1$, when $f(z)$ is schlicht and star-like with respect to the origin, then $\rho_1=2\pi$ and $|b_n|\leqq 2$ [R. Nevanlinna, Översikt Finska Vet.-Soc. Förhandlingar 63, no. 6 (1921)].

Let $x$ be the reciprocal of the radius of the smallest circle which contains the image of the unit circle given by the schlicht function $f(z)$, normalized as in (1). Löwner's differential equation with parameter $t$, $0\leqq t\leqq t_0$, $x=e^{-t_0}$, is derived for the slit functions $f(z, u)$, $u=e^{-t}$, $x\leqq u\leqq 1$, approximating $e^{-t_0}f(z)$. Letting $w=F(z, u)=u^{-1}f(z, u)$, and denoting by $z=\phi(w, u)$ the inverse function of $F(z, u)$, the author obtains the Löwner differential equation for $\phi(w, u)$:

$$(2) \qquad \frac{\partial\phi}{\partial u}(w, u)=-2w\frac{\partial\phi(w, u)}{\partial w}\cdot\frac{e^{i\theta(u)}}{1-ue^{i\theta(u)}w}$$

where $\theta(u)$ is continuous and real in $x\leqq u\leqq 1$. Let

$$\frac{w}{\phi}\,\frac{\partial\phi(w, u)}{\partial w}=1+\sum_{n=1}^{\infty}\beta_n(u)w^n.$$

From (2) the sharp bounds for $\beta_n(u)$ are obtained, so that in the limiting case the corresponding $\beta$ coefficients associated with the inverse function of $f(z)$ satisfy the sharp inequalities

$$|\beta_n(x)|\leqq\sum_{\nu=0}^{n}(-1)^\nu\frac{2n}{\nu!}\,\frac{(2n-\nu-1)!}{((n-\nu)!)^2}x^\nu\leqq\frac{(2n)!}{(n!)^2}, \quad n=1, 2, \cdots.$$

Similarly, sharp bounds are obtained for the coefficients of the inverse function of $f(z)$ in terms of the parameter $x$. For $x=0$ these bounds agree with those obtained earlier by Löwner [Math. Ann. 89, 103–121 (1923)].

For $f(z)$ normalized and schlicht in

$$|z|<1, \quad |f(z)|\leqq R=1/x,$$

it is shown that the coefficients $a_3$ and $b_2$ of (1) satisfy

$$\begin{array}{lll} (3) & |a_3(x)|\leqq 1-x^2 & \text{for} \quad e^{-1}\leqq x\leqq 1, \\ & |a_3(x)|\leqq 3-8x+7x^2 & \text{for} \quad 0\leqq x\leqq \tfrac{1}{2}, \\ (4) & |b_2(x)|\leqq 2(1-x^2) & \text{for} \quad e^{-2}\leqq x\leqq 1, \\ & |b_2(x)|\leqq 3-6x+7x^2 & \text{for} \quad 0\leqq x\leqq \tfrac{1}{3}. \end{array}$$

For the regions of variability $R(a_{n+1}(x))$, $R(b_n(x))$ it is shown that their boundary curves are non-ascending in the interval $0\leqq x\leqq 1$.

The author shows that if $\theta(u)$ is replaced by a step-function $\theta_p(u)$ with a finite number of discontinuities, the resulting solution $F_p(z, x)$ of Löwner's equation is also schlicht. The subclass $S_p$ of schlicht functions obtained in this way can be exhausted by the class $S_L$ of Löwner slit functions, and conversely. It is shown that (3) and (4) are sharp with equality signs holding for certain functions of $S_p$ with one discontinuity. If $\theta(u)$ is a step-function with but one discontinuity, expressions are derived for $a_4$, $a_5$, $b_3$, $b_3$, and $b_4$ and it is found that the Koebe function maximalizes $|a_4|$ and $|a_5|$ in this class, and all the $b$-coefficients in question exceed 2.

*M. S. Robertson.*

**Buckel, Walter. Ähnlichkeit im Grossen bei konformen Abbildungen.** S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 163–189 (1952).

The author says a mapping $w=f(z)$ of a set $M$ has similarity in the large if for any two points $z_1$ and $z_2$ of $M$ the expression $|f(z_1)-f(z_2)|/|z_1-z_2|=d$ is a constant different from zero. In a region $G$ conditions are developed for a conformal mapping to transform with similarity in the large a proper subset of $G$ which is dense in itself.

*P. R. Garabedian* (Stanford, Calif.).

**Albrecht, Rudolf. Zum Schmiegungsverfahren der konformen Abbildung.** Z. Angew. Math. Mech. 32, 316–318 (1952).

In Koebe's proof of the Riemann mapping theorem [J. Reine Angew. Math. 145, 177–223 (1915)], a sequence of functions is constructed whose limit is the function which maps a given simply connected region conformally on the unit circle. The author proposes replacing Koebe's functions by another class of functions for which the convergence will be more rapid, and using this procedure to approximate the mapping function. This class of functions includes as special cases, the transformations suggested by Ringleb [National Research Council of Canada, Division of Mechanical Engineering, Technical Translation no. TT-70 (1948); these Rev. 11, 341], and by Heinhold [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1948, 203–222; these Rev. 11, 507].

*C. Saltzer* (Cleveland, Ohio).

**Bonder, Julian. Sur les fonctions réalisant les représentations conformes et biunivoques d'un demi-plan sur les extérieurs des arcs de certaines courbes algébriques.** Czechoslovak Math. J. 1(76) (1951), 203–228 (1952) = Čehoslovack. Mat. Ž. 1(76) (1951), 229–257 (1952).

Let $\alpha$ be a given simple analytic arc in the complex $z$-plane such that there exists an $n$-valued algebraic function

$\zeta=\varphi(z)$ all of whose branches map $\alpha$ onto circular arcs. This paper is concerned with the investigation of the function $z=f(t)$ which maps the schlicht upper half-plane $\mathrm{Im}\{t\}>0$ onto the complement of $\alpha$. The author proves that the Schwarzian derivative of the composite function $\varphi[f(t)]$ is again an $n$-valued algebraic function, say $F(t)$. If the Riemann surface on which $\varphi(z)$ is single-valued is of genus zero, the Riemann surface $R$ of $F(t)$ is of genus one and $f(t)$ can be expressed in terms of elliptic functions if it is possible to determine the branch-points of $R$ and the poles and residues of $F(t)$; in all other cases the genus of $R$ is $\geqq 3$ and this, for all practical purposes, excludes an effective computation of $f(t)$. The author illustrates his theory by carrying out the computations in the cases in which $\alpha$ is an arc of a conic or of a Cassinian. The reviewer wishes to point out that all these mappings can be constructed much more simply by elementary direct methods (see the reviewer's "Conformal mapping" [McGraw-Hill, New York, 1952, Chap. VI, sec. 4; these Rev. 13, 640]). *Z. Nehari* (St. Louis, Mo.).

**Goodman, A. W. Inaccessible boundary points.** Proc. Amer. Math. Soc. 3, 742–750 (1952).

Exemples effectifs de fonctions représentant conformément le cercle $|z|<1$ sur un domaine possédant des points frontières inaccessibles. Les fonctions considérées sont de la forme

$$F(z)=\frac{cz}{\prod_{n=1}^{\infty}(1-2z\cos\theta_n+z^2)^{\gamma_n}},$$

$$\sum_{n=1}^{\infty}\gamma_n=1,\quad \gamma_n>0,\quad \pi>\theta_1>\cdots>\theta_n>0,\quad \theta_n\to 0.$$

*J. Lelong* (Lille).

**Consiglio, A. Sopra la rappresentazione conforme di un piano con foro a contorno regolare in uno con foro a contorno circolare.** Ricerca, Napoli 2, no. 1, 66–68 (1951); 3, no. 2, 16–25 (1952).

L'auteur étudie l'application conforme ($\zeta\to z$) du domaine extérieur à une courbe simple de Jordan sur le domain extérieur à un cercle avec correspondance des points à l'infini. Laissant de coté les conditions de validité des développements qu'il introduit, il donne un développement formel $z=A+B\zeta+\sum k_n/\zeta^n$ où les coefficients s'expriment à l'aide de ceux de certains développements de Fourier relatifs à la correspondance des frontières. Application à quelques cas simples. *M. Brelot* (Grenoble).

**Meschkowski, Herbert. Einige Extremalprobleme aus der Theorie der konformen Abbildung.** Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 117, 12 pp. (1952).

The author discusses in detail familiar extremal characterizations of canonical conformal mappings of a multiply-connected domain and develops representations of such mappings in terms of an orthonormal system which are related to the representations by the kernel function.

*P. R. Garabedian* (Stanford, Calif.).

**Wittich, Hans. Über eine Extremalaufgabe der konformen Abbildung.** Arch. Math. 2, 325–333 (1950).

Using the length-area principle, the author discusses the problem of maximizing the modulus of a ring domain whose two boundary components include given points.

*P. R. Garabedian* (Stanford, Calif.).

**Wittich, Hans. Bemerkung zum Typenproblem.** Comment. Math. Helv. 26, 180–183 (1952).

The author derives a relation between his well-known type criterion [Math. Z. 45, 642–668 (1939); these Rev. 1, 211] and a criterion of the reviewer [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 50, (1948); these Rev. 10, 365]. The former deals with simply connected Riemann surfaces whose branch points lie over a finite number of points of the complex plane. It asserts that, if $\sigma(n)$ is the number of knots in the first $n$ generations of the line complex, then the condition $\sum 1/\sigma(n)=\infty$ is sufficient in order that the type be parabolic. The reviewer's criterion states that, for an arbitrarily given Riemann surface, the divergence of the modular product $\prod\mu_n$ of an exhaustion guarantees that the surface be of parabolic type. The author shows that, in the absence of algebraic branch points, the relations

$$c_1\sum(1/\sigma(n))\leqq\log\prod\mu_n\leqq c_2\sum(1/\sigma(n))$$

hold for an exhaustion suitably associated with the line complex. *L. Sario* (Cambridge, Mass.).

**Radojčić, M. Sur les singularités essentielles de certaines fonctions automorphes dans un domaine.** Acad. Serbe Sci. Publ. Inst. Math. 4, 129–132 (1952).

The author gives a number of applications of his theorem on "fundamental regions" [same Publ. 3, 137–142 (1950); these Rev. 12, 690]. For suitably restricted classes of automorphic functions and for certain decompositions of the regions of existence, the diameters of the fundamental regions with a converging sequence of points tend to zero.

*L. Sario* (Cambridge, Mass.).

**Nagai, Yasutaka. On the behaviour of the boundary of Riemann surfaces. I.** Proc. Japan. Acad. 26, nos. 2–5, 111–115 (1950).

**Nagai, Yasutaka. On the behaviour of the boundary of Riemann surfaces. II.** Proc. Japan Acad. 26, no. 6, 10–16 (1950).

The author reproves some well-known theorems concerning the boundary of a parabolic covering surface of the complex plane. He also extends a number of results to subregions with a compact relative boundary and with an ideal boundary of harmonic measure zero. A typical theorem asserts that, if a point is covered infinitely many times, then so is every point of the complex plane, except for a set of vanishing capacity. *L. Sario* (Cambridge, Mass.).

**Mori, Akira. On Riemann surfaces, on which no bounded harmonic function exists.** J. Math. Soc. Japan 3, 285–289 (1951).

Let $F$ be a Riemann surface with $HB$-removable boundary, i.e., not admitting harmonic bounded single-valued non-constant functions. Consider $F$ to be a covering surface of the complex $z$-plane. The part of $F$ lying above a disc $K$: $|z-z_0|<\rho$ consists of a finite or infinite number of disjoint regions $F_\nu$. The author proves that each $F_\nu$ covers every point of $K$, except possibly a set of logarithmic capacity zero. The proof is based on Frostman's theorem [Medd. Lunds Univ. Mat. Sem. 3 (1935)] on Seidel's functions [Trans. Amer. Math. Soc. 36, 201–226 (1934)].

Denote by $n(z)$, $0\leqq n(z)\leqq\infty$, the number of sheets of $F_\nu$ above $z\in K$. It follows from the preceding theorem that the set of points $z$ with $n(z)<\sup n(z)\leqq\infty$ is of (inner) capacity zero. Earlier it was shown by Nagai [see the second paper reviewed above] and Tsuji [Kōdai Math. Sem. Rep. 1950, 89–93; these Rev. 13, 125] that every parabolic surface has this property. *L. Sario* (Cambridge, Mass.).

**Sommer, Friedrich. Über die Integralformeln in der Funktionentheorie mehrerer komplexer Veränderlichen.** Math. Ann. **125**, 172–182 (1952).

In the space of $k$ complex variables $z^1, \cdots, z^k$ consider several holomorphic functions $w^\nu=\phi_\nu(z^1, \cdots, z^k)$, and for each consider in $E_{2k}$ the pre-image of the map generated by it into a two-dimensional domain in the $w^\nu$-plane whose boundary is formed by analytic arcs. Denote the intersection of the pre-images by $D$ and assume that it is a simplicial $2k$-complex whose boundary $B$ has certain analytic regularities; and denote by $B_{k+m}$, $0\leqq m\leqq k-1$, the $(k+m)$-dimensional "skeleton" of $B$. Now, for $\zeta$ on $B$ and $z$ not on $B$ there is a form $K_{a_1\cdots a_m}(\zeta, \bar\zeta; z, \bar z)d\bar\zeta^{a_1}\cdots d\bar\zeta^{a_m}$ such that for a function $f(z)$ which is holomorphic in $D+B$ we have

$$\int_{B_{k+m}} f(\zeta)K_{a_1\cdots a_m}(\zeta, \bar\zeta; z, \bar z)d\bar\zeta^{a_1}\cdots d\bar\zeta^{a_m}d\zeta^1\cdots d\zeta^k=f(z)$$

for $z$ in $D$, and $=0$ for $z$ in $E_{2k}-(D+B)$. For $m=0$ this is a formula of A. Weil; for $m=k-1$ it is a formula of Martinelli and also the reviewer; and for $0\leqq m\leqq k-2$ (including $m=0$) the author obtains it from the latter by a reduction of dimension. It should be noted, however, that only for $m=k-1$ is the form under the integral structurally independent of the domain as in the classical case when it is $(\zeta-z)^{-1}d\zeta$. There is no mention of applications. *S. Bochner.*

**Mitchell, Josephine. The kernel function in the geometry of matrices.** Duke Math. J. **19**, 575–583 (1952).

Let $Z=(z_{jk})$ be an $m$ by $n$ matrix with complex elements, $Z^{*\prime}$ the conjugate transpose of $Z$, and let $ZZ^{*\prime}<I$ designate the set of all points $Z$ with $mn$ complex coordinates $z_{jk}$ for which the quadratic form $\sum_{j,k=1}^{m}(\delta_{jk}-\sum_{i=1}^{n}z_{ji}z^*_{ki})u_ju_k^*$ is positive definite. The author shows that the Bergman kernel function of the domain $D$: $ZZ^{*\prime}<I$ is given by $V^{-1}[\det(I-ZZ^{*\prime})]^{-p}$. Here $V$ is the Euclidean volume of $D$ and $p=m+n$ for rectangular matrices, $n+1$ for symmetric, and $n-1$ for skew-symmetric matrices. The singular behavior of the kernel function of $D$ in the neighborhood of boundary points is also studied. *P. Davis.*

**Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. Some properties in matrix space.** Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 230–237 (1952).

Let $W\equiv(w_{ij})$ be an $(m_1\times n_1)$ matrix and $Z\equiv(z_{ij})$ an $(m_2\times n_2)$ one. The matrix $W$ is called an analytic function of $Z$ if each element $w_{ij}$ of $W$ is an analytic function of all the elements $z_{ij}$ of $Z$. Let $R$ be the totality of all $n$-rowed square matrices whose elements are complex numbers. For $W$ and $Z$ in $R$, the authors call an analytic function $W=W(z)$ "univalent" in $R$ if for any pair of variables $Z^{(1)}\neq Z^{(2)}$ in $R$ the inequality $w_{ij}(Z^{(1)})\neq w_{ij}(Z^{(2)})$ holds for at least one element $w_{ij}$ of $W$. The authors prove two theorems giving sufficient conditions for such a function to be "univalent". As their first theorem, they prove that an analytic function $W=W(Z)$ in a convex domain $K$ of $R$ is univalent in $K$ if there exists a matrix $A\equiv(a_{ij})$ belonging to $R$ with $\det A\neq 0$ for which

$$\sum_{i,j,k,l=1}^{n}\left|\frac{\partial w_{ij}}{\partial z_{kl}}-\delta_{jl}a_{ik}\right|^2<\lambda$$

holds throughout $K$ where $\lambda$ is the minimum of the characteristic values of $A^*A$, $\delta_{jl}$ is Kronecker's delta and $A^*$ is the adjoint matrix of $A$. The proof uses several properties of differentials in matrix space which the authors develop, as well as lemmas on norms of matrices. By suitably specializing the theorem, more specifically by setting $z_{ij}=\delta_{ij}z_i$, $w_{ij}=\delta_{ij}w_i$, the authors obtain as a corollary a result due to S. Takahashi [Ann. of Math. (2) **53**, 464–471 (1951); these Rev. **12**, 818] on "univalent" mappings in several complex variables. *W. T. Martin* (Cambridge, Mass.).

**Saxer, Walter. Errata.** J. Math. Pures Appl. (9) **31**, 380 (1952).

See same J. **31**, 49–53 (1952); these Rev. **13**, 834.

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## *Theory of Series*

**Hyslop, J. M. Note on the strong summability of series.** Proc. Glasgow Math. Assoc. **1**, 16–20 (1952).

A series $\sum a_n$ with Cesàro transform $C_n^{(k)}$ of positive order $k$ is said to be strongly evaluable $(C, k)$ with index $p$, or to be evaluable $[C, k, p]$ to $s$ if $n^{-1}\sum_{\nu=0}^{n}|C_\nu^{(k-1)}-s|^p\rightarrow 0$ as $n\rightarrow\infty$. If $p\geqq 1$, then $\sum a_n$ is evaluable $[C, k, p]$ if and only if it is evaluable $(C, k)$ and $n^{-1}\sum_{\nu=0}^{n}\nu^p|C_\nu^{(k)}-C_{\nu-1}^{(k)}|\rightarrow 0$ as $n\rightarrow\infty$. If $p\geqq 1$ and $\sum a_n$ is evaluable $[C, k, p]$, then $\sum a_n$ is evaluable $[C, k+h, q]$ when $h>0$ and $0<q\leqq p$. *R. P. Agnew.*

**Petersen, G. M. A note on divergent series.** Canadian J. Math. **4**, 445–454 (1952).

The first part of the paper treats the Rogosinski-Bernstein transformation

$$B_n^{(h)}=\sum_{k=0}^{n}u_k\cos\frac{k\pi}{2(n+h)}$$

of order $h$, $0<h<1$, by which $\sum u_k$ is evaluable to $s$ if $B_n^{(h)}\rightarrow s$ as $n\rightarrow\infty$. It is shown that if $\frac{1}{2}<h<1$, then $B^{(h)}$ is equivalent to the Cesàro method $C_1$. The question whether $B^{(h)}$ and $C_1$ are equivalent when $0<h<\frac{1}{2}$ is raised but left unanswered. As is shown in a contemporary paper by the reviewer [see the paper reviewed below], $B^{(h)}$ is stronger than $C_1$ when $0<h<\frac{1}{2}$.

The remainder of the paper treats chopped-off Nörlund transformations $N_\alpha$ of the form

$$\sigma_n=\alpha_0 s_{n-p}+\alpha_1 s_{n-p+1}+\cdots+\alpha_p s_n$$

where $\alpha_0+\alpha_1+\cdots+\alpha_p=1$. The main result is a characterization of the class of sequences $s_n$ evaluable $N_\alpha$. This characterization involves the zeros of the associated polynomial $\alpha_0+\alpha_1 z+\cdots+\alpha_p z$. It is used to prove the following Tauberian-Mercerian theorem. In order that $N_\alpha$ be such that each bounded sequence evaluable $N_\alpha$ is convergent, it is necessary and sufficient that each zero of the associated polynomial be either inside or outside the unit circle $|z|=1$. This complements the Mercerian theorem of Kubota [Tôhoku Math. J. **12**, 222–224 (1917)] which says that each sequence evaluable $N_\alpha$ is convergent if and only if each zero of the associated polynomial lies inside the unit circle.

*R. P. Agnew* (Ithaca, N. Y.).

**Agnew, Ralph Palmer. Rogosinski-Bernstein trigonometric summability methods and modified arithmetic means.** Ann. of Math. (2) **56**, 537–559 (1952).

Karamata [Mat. Sbornik N.S. **21**(63), 13–24 (1947); these Rev. **9**, 140] considered methods of summation $B_h$, $0<h\leqq 1$, given by the transformation $\sigma_n=\sum_{k=0}^{n}\cos(k\pi/2(n+h))u_k$ and stated that $B_h$ is equivalent to $(C, 1)$ for $|h-\frac{1}{2}|\geqq c$ with some constant $c<0.05$. The reviewer pointed out [Zentralblatt für Math. **29**, 208 (1948)] that the proof is not correct but may be made conclusive for $\frac{1}{2}+c\leqq h\leqq 1$. Karamata also

proved [Math. Z. 52, 305–306 (1949); these Rev. 11, 347] that $B_{1/2}$ is equivalent to the method defined by $\sigma_n=\frac{1}{2}(M_{n-1}+M_n)$, where $M_n=(n+1)^{-1}\sum_{k=0}^{n}s_k$. The main results of the author are: (1) $B_h$ is equivalent to $(C, 1)$ for $\frac{1}{2}<h\leqq 1$ [this has been proved independently by Petersen, see the paper reviewed above]; (2) $B_h$ is stronger than $(C, 1)$ for $0<h\leqq\frac{1}{2}$; (3) each sequence which is $B_h$ and $B_{h'}$ summable with $h\neq h'$, $h\leqq\frac{1}{2}$, $h'\leqq\frac{1}{2}$, is $(C, 1)$ summable. Further results concern the methods $M^{(h)}$, $M^{*(h)}$ defined by

$$\sigma_n=(1-h)n(n+1)^{-1}M_{n-1}+hM_n \text{ and } \sigma_n=(1-h)M_{n-1}+hM_n$$

respectively. These methods are compared with each other and with the methods $B_h$. For example, if $0<h\leqq\frac{1}{2}$ and $0<H\leqq\frac{1}{2}$, then $B_h$ can include $M^{(H)}$ and $M^{*(H)}$ only if $h=H=\frac{1}{2}$. *G. G. Lorentz* (Toronto, Ont.).

**Wilansky, Albert. Summability: the inset, replaceable matrices, the basis in summability space.** Duke Math. J. 19, 647–660 (1952).

Let $A=(a_{nk})$ be a matrix defining a method of summation. The following definitions are given. If $a_k=\lim_n a_{nk}$, the inset $[A]$ of $A$ is the set of all sequences $s=(s_n)$ for which $s \in (A)$ (that is, $s_n$ is $A$-summable) and $\sum a_k s_k$ converges. A topology may be defined on $(A)$. If $[A]=(A)$, $A$ has maximal inset. $A$ has the PMI (propagation of the maximal inset) property if each matrix $B$ with $(B)=(A)$ has maximal inset. $A$ is replaceable if there is a regular matrix $B$ with $(B)=(A)$. These notions are used to study $(A)$ and $[A]$ by means of properties of the matrix $A$. As examples we quote: (1) A regular method $A$ with $a_{nk}=s_n t_k$, $k\leqq n$, $a_{nk}=0$ for $k>n$ has the PMI property, while the method defined by $\sigma_n=(s_n+s_{n+1})/2$ has not. (2) Let $a_{nn}\neq 0$, $a_{nk}=0$ for $k>n$ and $A$ be co-regular, that is, $\lim_n \sum_k a_{nk}-\sum a_k\neq 0$. Then $A$ has the PMI property if and only if the set $(1, 1, 1, \cdots)$, $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots)$, $\cdots$ is a basis for $(A)$. (3) There is a matrix $A$ of this type such that no matrix $B$ with $(B)\supset(A)$ has maximal inset. (4) A co-regular matrix with maximal inset is replaceable, but there are co-regular non-replaceable matrices. *G. G. Lorentz* (Toronto, Ont.).

**Tolba, S. E. On the summability of Taylor series at isolated points outside the circle of convergence.** Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 380–387 (1952).

Let $f(z)=\sum c_n z^n$ be a Taylor series with radius of convergence 1 and let $z_1, \cdots, z_m$ be given regular points of $f(z)$ with $|z_k|>1$. If the $c_n$ and the $z_k$ satisfy certain conditions, there is a regular matrix method of summation which sums $\sum c_n z^n$ (to the value $f(z)$) outside of $|z|=1$ at exactly $mp$ points $z_k \exp(2\pi li/p)$, $k=1, \cdots, m$, $l=0, \cdots, p-1$. [Reviewer's remark: Darevsky [Izvestiya Akad. Nauk SSSR. Ser. Mat. 10, 97–104 (1946); these Rev. 7, 517] proved the existence of a regular method summing exactly the sequences $as_n+t_n$ with an arbitrary unbounded $s_n$ and convergent $t_n$. The question under which conditions the corresponding statement is true for $m$ given sequences seems to be open.] *G. G. Lorentz* (Toronto, Ont.).

**Gaier, Dieter. Über die Summierbarkeit beschränkter und stetiger Potenzreihen an der Konvergenzgrenze.** Math. Z. 56, 326–334 (1952).

If $f(z)=\sum_{n=0}^{\infty}a_n z^n$ is bounded in $|z|<1$, the methods of summation $E_p$, $0<p<+\infty$, $B$, $S_\alpha$, $0<\alpha<1$, $T_\alpha$, $0<\alpha<1$, are equivalent if applied to $\sum a_n$. [For the methods $S_\alpha$, $T_\alpha$ see Meyer-König [Math. Z. 52, 257–304 (1949); these Rev. 11, 242].] The proof is based on the estimate $b_n=O(n^{-1/2})$ for a series $\sum b_n z^n$ regular and bounded in $|z+a|<1+a$, $a>0$, and the Tauberian theorem of the methods $T_\alpha$. A corresponding statement is true for the methods $(C, k)$, $0<k<+\infty$, and L. Fejér [S.-B. Bayer. Akad. Wiss. Math.-Phys. Kl. 1910, no. 3] constructed a function $f(z)=\sum a_n z^n$ continuous for $|z|\leqq 1$ for which $\sum a_n$ diverges. The author shows that in this example $\sum a_n$ is not $B$-summable, so that the above two groups of methods prove to be different even for series $\sum a_n z^n$ continuous in $|z|\leqq 1$. *G. G. Lorentz.*

**Paterson, S. The summation of a slowly convergent series.** Edinburgh Math. Notes 38, 5–7 (1952).

Formulas are obtained for estimating the value of $\sum_{n=1}^{\infty}n^{2r}e^{-n^2x}$ when $r$ is a positive integer and $x$ is positive and near zero. *R. P. Agnew* (Ithaca, N. Y.).

**Emersleben, Otto. Über die Reihe**

$$\sum_{k=1}^{\infty}\frac{k}{(k^2+c^2)^2}.$$

**(Bemerkung zu einer Arbeit von Herrn K. Schröder.)** Math. Ann. 125, 165–171 (1952).

The series in the title converges to a function $s(c)$ about which information is needed in applied mathematics. It is shown that $c^2 s(c)$ is increasing over $c>0$, and that

$$\frac{1}{2}-\frac{5}{32}\frac{1}{c^2}\leqq c^2 s(c)<\frac{1}{2}$$

when $c\neq 0$. It follows that $\frac{1}{2}$ is the least constant $h$ such that $s(c)<hc^{-2}$ when $c\neq 0$. *R. P. Agnew* (Ithaca, N. Y.).

**James, Robert C. Infinite series and Taylor and Fourier expansions.** Math. Mag. 25, 269–272; 26, 21–31 (1952).

## *Fourier Series and Generalizations, Integral Transforms*

**Zarantonello, Eduardo H. On trigonometric interpolation.** Proc. Amer. Math. Soc. 3, 770–782 (1952).

Let $P(t)$ be a trigonometrical polynomial of degree $n$, and $P^{(r)}(t)$ and $P^{[r]}(t)$ be the $r$th derivative and the $r$th difference quotient respectively, i.e., $P^{[r]}(t)=\Delta^r P(t)/(\Delta t)^r$ with $\Delta t\leqq 2\pi/(2n+1)$. Let $w_n(t)$ be $2k\pi/n$ in the interval $(2k\pi/n, 2(k+1)\pi/n)$ for $k=0, 1, 2, \cdots, n-1$. The author proves that, for $1<p<\infty$,

$$\left(\int_0^{2\pi}|P^{(r)}(t)|^p dt\right)^{1/p}\leqq A_{p,r}\left(\int_0^{2\pi}|P^{[r]}(t)|^p dt\right)^{1/p},$$

and that $dt$ in the right side integral may be replaced by $dw_{2n+1}$. The case $r=0$ was proved by Marcinkiewicz [Studia Math. 6, 1–17, 67–81 (1936)]. As an application, he proves that: if $f(t)$ is a periodic function of period $2\pi$ with absolutely continuous derivatives up to the $(r-1)$th order and $P_n(t)$ is a trigonometrical polynomial of degree $n$ coinciding with $f(t)$ at more than $2n$ equidistant points in $(0, 2\pi)$, then

$$\left(\int_0^{2\pi}|P_n^{(r)}(t)|^p dt\right)^{1/p}\leqq D_{p,r}\left(\int_0^{2\pi}|f^{(r)}(t)1^p dt\right)^{1/p},$$

$$|f(t)-P_n(t)|\leqq\frac{C_{p,r}}{n^{r-1/p}}\left(\int_0^{2\pi}|f^r(t)|^p dt\right)^{1/p},$$

$$\left(\int_0^{2\pi}|f(t)-P_n(t)|^p dt\right)^{1/p}\leqq\frac{D_{p,r}}{n^r}\left(\int_0^{2\pi}|f^{(r)}(t)|^p dt\right)^{1/p}$$

for $1<p<\infty$. The author proves further the analogues of above inequalities for the cases $p=0$ and $p=\infty$, and remarks on the case $0<p<1$. *S. Isumi* (Tokyo).

**Prasad, B. N., and Singh, U. N. On the strong summability of the derived Fourier series and its conjugate series.** Math. Z. **56**, 280–288 (1952); corrigenda and addenda, **57**, 481–482 (1953).

Let $f(t)$ be continuous and of bounded variation on $[0, 2\pi]$, and let $t=x$ be a point such that $f'(x)$ exists. If $g(t)=f(x+t)-f(x-t)-2f'(t)$, the authors establish that $\int_0^t |dg(u)|=o\{t/(\log t^{-1})^{1+\epsilon}\}$ as $t\to 0$ is sufficient for the $|C, 1|$-summability of the derived Fourier series of $f(t)$ at $t=x$. The insufficiency of the continuity of $f'(t)$ at $t=x$ for the same conclusion has been established by Singh [Proc. Nat. Acad. Sci. India. Sect. A. **15**, 63–72 (1946)]. A corresponding result is also established for the series conjugate to the derived Fourier series. *P. Civin* (Eugene, Ore.).

**Misra, M. L. The summability of the successively derived series of a Fourier series by Riesz's logarithmic means.** Saugar Univ. J. **1**, 197–207 (1952).

Let $f(\theta)$ be Lebesgue integrable over $(-\pi, \pi)$ and of period $2\pi$. Let $g(t)=[f(\theta+t)-P(t)+(-1)^r\{f(\theta-t)-P(-t)\}]/2t^r$, $0\leqq t<\pi$, where $P(t)=\sum_{p=0}^{r-1}\alpha_p t^p/p!$. The following results are established. If $\int_0^t |g(t)-r!A|\,dt=o(t)$ as $t\to 0$, then the $r$th derived Fourier series of $f(\theta)$ is summable $(R, \log n, r)$ to the value $A$ at the point $\theta$. Secondly, if $P(t)$ is such that $g(t)$ is Lebesgue integrable over $[0, \pi]$, then a necessary and sufficient condition that the $r$th derived Fourier series of $f(\theta)$ be summable $(R, \log n, r+\delta)$, $\delta\geqq 0$, to the sum $r!S$ at the point $\theta$ is that the Fourier series of $g(t)$ be summable $(R, \log n, \delta)$ to the sum $S$ for $t=0$. *P. Civin.*

**Alexits, Georges. Sur l'ordre de grandeur de l'approximation d'une fonction périodique par les sommes de Fejér.** Acta Math. Acad. Sci. Hungar. **3**, 29–42 (1952). (Russian summary)

It is well known that if $0<\alpha<1$, the necessary and sufficient condition for the periodic function $f(x)$ to belong to the class Lip $\alpha$ is that $|f(x)-\sigma_n(x)|=O(n^{-\alpha})$ uniformly, where $\sigma_n(x)$ is the Fejér sum of order $n$ of the Fourier series of $f$. For $\alpha=1$ the result is no longer true, but it was shown by the author in a previous paper [Mat. Fiz. Lapok **48**, 410–422 (1941); these Rev. **8**, 261] that $f\in$ Lip 1 if and only if $|\bar{f}-\bar{\sigma}_n|=O(n^{-1})$, where $\bar{f}$ is the conjugate of $f$ and $\bar{\sigma}_n$ is its Fejér sum. The main result of the present paper is a unification and a generalization of these results in the following theorem: Let $0<\alpha\leqq 1$ and $1\leqq p\leqq\infty$; then $f\in \mathrm{Lip}(\alpha, p)$ if and only if $\|\bar{f}-\bar{\sigma}_n\|_p=O(n^{-\alpha})$ where $\|\ \|_p$ denotes the norm in the metric $L^p$. The case $\alpha=p=1$ corresponds, as is known from Hardy and Littlewood, to functions equal almost everywhere to functions of bounded variation. The paper contains other results pertaining to the order of approximation at almost every point. *R. Salem.*

**Szász, Otto. On the Gibbs phenomenon for a class of linear transforms.** Acad. Serbe Sci. Publ. Inst. Math. **4**, 135–144 (1952).

Let $\varphi(t)$ be a function defined and of bounded variation in $(0, \infty)$, such as $\varphi(+0)=\varphi(0)=1$. Consider the transform $T(h, t)=\sum_{n=1}^{\infty}\varphi(nh)\sin nt/n$ and put

$$G=\limsup_{h\to 0,\, t\to 0} 2T(h, t)/\pi \quad (\geqq 1).$$

The author says that the transform $T$ presents a Gibbs phenomenon when $G>1$, which coincides with the ordinary definition for summability of Fourier series, taking $\varphi(t)$ as its kernel. He proves the formula

$$G=\frac{2}{\pi}\max_{0<\zeta\leqq\infty}\int_0^{\infty}\varphi(t)\frac{\sin \zeta(t)}{t}dt$$

and applies it to the summation of Riesz, Lambert, etc., taking

$$\varphi(t)=(1-t^{\lambda})^k \quad (0\leqq t\leqq 1), \qquad \varphi(t)=0 \quad (t\geqq 1);$$
$$\varphi(t)=te^{-t}/(1-e^{-t}) \quad (t>0), \qquad \varphi(0)=1;$$

etc. [cf. Szász, Acta Sci. Math. Szeged **12B**, 107–111 (1950); Trans. Amer. Math. Soc. **69**, 440–456 (1950); these Rev. **11**, 658; **12**, 405; B. Kuttner, J. London Math. Soc. **19**, 153–161 (1944); **20**, 136–139 (1945); these Rev. **7**, 154, 518]. *S. Isumi* (Tokyo).

**Mitchell, Josephine. On the spherical convergence of multiple Fourier series.** Amer. J. Math. **73**, 211–226 (1951); correction, **75**, 57–59 (1953).

This paper gives a sufficient condition for the convergence, almost everywhere, of multiple Fourier series summed over spheres. If $\{a_{n_1\cdots n_k}\}$ are the Fourier coefficients, the condition is:

$$\sum(n_1^2+\cdots+n_k^2)^p|a_{n_1\cdots n_k}|^2<\infty,$$

where $p=\frac{1}{2}$ for $k=2$, $p=k/4$ for $k\geqq 4$ and the sequence $(n_1^2+n_2^2+n_3^2)^p$ is replaced by

$$(n_1^2+n_2^2+n_3^2)^{2/3}(\log(n_1^2+n_2^2+n_3^2))^{5/3}$$

if $k=3$. The proof is based on an estimate of the integral

$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|\varphi_R(\alpha-x)|\,dx_1\cdots dx_k$$

where

$$\varphi_R(\alpha-x)=(2\pi)^{-k}\sum_{\nu\leqq R}\exp(n_1(\alpha_1-x_1)+\cdots+n_k(\alpha_k-x_k))$$

and $\nu^2=n_1^2+\cdots+n_k^2$. Once this estimate has been made, the argument proceeds on classical lines. *K. Chandrasekharan* (Bombay).

**Levin, B. Ya. On the almost periodic functions of Levitan.** Ukrain. Mat. Žurnal **1**, no. 1, 49–101 (1949). (Russian)

For Bohr almost periodic functions there exist several equivalent definitions and the fundamental theorems are concerned with their equivalence. The author, following these ideas, does the corresponding work for the recent generalization of almost periodic functions given by Levitan [Uspehi Matem. Nauk (N.S.) **2**, no. 5(21), 133–192; no. 6(22), 174–214 (1947); these Rev. **10**, 292, 293]. $\tau$ is called an $(\epsilon, N)$ translation of $f$ if $|f(t+\tau)-f(t)|<\epsilon$ for $|t|<N$. A continuous function $f$ is called Lap(Levitan almost periodic) if (i)for every $\epsilon>0$ and $N>0$, there exists a set $E_{\epsilon N}$ of $(\epsilon, N)$ translations which is relatively dense and (ii) for every $(\epsilon, N)$, there exists a $\delta>0$ such that $E_{\delta N}\pm E_{\delta N}\subseteq E_{\epsilon N}$. The author shows the equivalence of the following definitions: He calls $\tau_n\to_f\tau_0$, if $\lim_n f(t+\tau_n-\tau_0)=f(t)$ pointwise for all $t$. This establishes a new topology for the real numbers and the equivalent definition of a Lap function is the following. A continuous function is Lap if and only if the real axis in the above topology becomes a conditionally compact group with addition a continuous operation. In the course of the equivalence proof, the author completes the above topological group and obtains a compact group. Using invariant integration the author can define a Fourier series and characteristic exponents of this Lap function. The un-

usual feature is that a Lap function can have different Fourier series, any one of which determines $f$. An example is constructed to exhibit a function with several Fourier series. The reason lies in the multiplicity of extensions of $f$ to the complete group. Next the author turns his attention to the module which corresponds to the module of characteristic exponents. Convergence with respect to a module is defined in the usual way. The class of Lap functions $\varphi$ whose module $M_\varphi$ is part of $M_f$ is identical with the functions continuous on the above defined incomplete group of real numbers. Approximation theorems of the following sort are proved. $f$ is Lap with modul $M$ if and only if for all $\epsilon, N>0$ there exist an $\eta>0$ and a trigonometric polynomial $P$ with characteristic exponents out of $M$, such that $|f(t+\tau)-P(t+\tau)|<\epsilon$ for $|t|<N$ and all $\delta$-translations $\tau$ of $P$. Bochner's summability of the Fourier series is established. It is now very easy to generalize Laperiodicity from the additive group of real numbers to an arbitrary group $G$. We shall say that $h_k \to_f e$, the identity of the group, if $\lim_{k\to\infty} \sup_{a\varepsilon G} |f(xah_ka^{-1})-f(x)|=0$. A continuous function $f$ on $G$ is called Lap if $G$ under the new topology becomes a conditionally compact (but in general not complete) topological group. The paper is provided with several appendices.

*František Wolf* (Berkeley, Calif.).

**Levin, B. Ya. On a generalization of the Fourier-Plancherel transform.** Učenye Zapiski Har'kov. Gos. Univ. **28**, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **20**, 83–94 (1950). (Russian)

The generalised Fourier transform is

$$\psi(\lambda)=\int_{-\infty}^{\infty} f(x)y(x,\lambda)dx$$

where $f$ is in $L^2(-\infty,\infty)$, the integral is a limit in mean, and $y(x,\lambda)=e^{ix\lambda}+G(x,\lambda)$ where $G(x,\lambda)$ is of summable square over the entire $(x-\lambda)$-plane. It is shown that if the formula (1) is one-one onto its range, then the inverse exists throughout $L^2(-\infty,\infty)$ and is of the same form, with kernel $z(x,\lambda)=e^{-ix\lambda}+G_1(x,\lambda)$ where $G_1$ is also of summable square. Necessary and sufficient conditions are given in order that a formula of Plancherel type

$$\int_{-\infty}^{\infty}|f(x)|^2d\sigma(x)=\int_{-\infty}^{\infty}|\psi(\lambda)|^2d\lambda$$

should hold for some monotonic $\sigma(x)$: if these hold, $\sigma(x)=x$.

If $y(x,\lambda)$ is an integral function of $x$ and $|y(x,\lambda)|<Me^{|x||\lambda|}$ for some $M$, then $z(x,\lambda)$ exists and satisfies the same conditions. Generalising a theorem of Paley and Wiener [Fourier transforms in the complex domain, Amer. Math. Soc. Colloq. Publ., v. 19, New York, 1934] it is shown that, under the same conditions on $y$, a necessary and sufficient condition that $\psi(\lambda)$ be an exponential function of type A is that $f(x)$ vanish outside $(-A,A)$.

If $y(x,\lambda)$ is the solution of the ordinary second order differential equation $Ly+\lambda^2y=0$, then $z$ is a solution of the adjoint equation $L^*z+\lambda^2z=0$. It is shown that all conditions are fulfilled by a solution of $y''+p(x)y+\lambda^2y=0$ if $xp(x)$ is in $L(-\infty,\infty)$ and is zero for $x<0$. *J. L. B. Cooper.*

**Berkovitz, Leonard D On double trigonometric integrals.** Trans. Amer. Math. Soc. **73**, 345–372 (1952).

F. Wolf [Univ. California Publ. Math. (N.S.) **1**, 159–227 (1947); these Rev. **9**, 140] and A. Zygmund [Ann. of Math. (2) **48**, 393–440 (1947); these Rev. **9**, 88] have proved the equiconvergence theorem: For a trigonometrical integral $\int_{-\infty}^{\infty} e^{ixu}d\varphi(u)$ where $\varphi(u)$ is of bounded variation in any finite interval and satisfies some regularity conditions, there is a continuous function $\alpha(u)$ such that

$$\int_{-\omega}^{\omega} e^{ixu}d\varphi(u)-\int_{-\omega}^{\omega} e^{ixu}\alpha(u)du\to 0 \quad (\omega\to\infty)$$

uniformly in any closed interval and there is a trigonometrical series $\sum a_ne^{inx}$ such that

$$\int_{-\omega}^{\omega} e^{ixu}d\varphi(u)-\sum_{n=-[\omega]}^{[\omega]} a_ne^{inx}\to 0 \quad (\omega\to\infty)$$

uniformly in any closed interval contained in $(0,2\pi)$. The author extends this theorem to double trigonometrical integrals, using restricted and circular convergence. He also extends the theory of the formal product, the multiplication theorem, and the localization theorem to double trigonometrical integrals. *S. Izumi* (Tokyo).

**Takeda, Ziro. A note on Fourier-Stieltjes integral.** Kōdai Math. Sem. Rep. **1952**, 59–61 (1952).

Let $G$ be a locally compact abelian group with elements $x, y, \cdots$ and let $\hat{G}$ denote its character group with elements $\hat{x}, \hat{y}, \cdots$. The author obtains two necessary and sufficient conditions that a measurable function $f(x)$ be representable almost everywhere in the form $f(x)=\int_{\hat{G}}(x,\hat{x})d\mu(\hat{x})$ where $\mu$ is a bounded measure on the Borel sets in $\hat{G}$. The first of these conditions generalizes a theorem due to I. J. Schoenberg for the case of the real line [Bull. Amer. Math. Soc. **40**, 277–278 (1934)] and requires that

$$\left|\int_G f(x)g(x)dx\right|\leqq M \sup_{\hat{x}\varepsilon\hat{G}}\left|\int_G (x,\hat{x})g(x)dx\right|$$

for all $g\varepsilon L_1(G)$. The second condition is somewhat deeper and requires that $|\sum_\alpha c_\alpha f(x_\alpha)|\leqq M \sup_{\hat{x}\varepsilon\hat{G}} |\sum_\alpha c_\alpha(x_\alpha,\hat{x})|$ for every finite set of $x_\alpha$'s and complex numbers $c_\alpha$. Bochner first proved this result for $f(x)$ continuous and $G$=real line [ibid. **40**, 271–276 (1934)]. This was later generalized by the reviewer [Trans. Amer. Math. Soc. **69**, 312–323 (1950); these Rev. **12**, 496] for $f(x)$ measurable on the real line and by M. Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 275–280 (1940); these Rev. **2**, 316] for $f(x)$ continuous on $G$.

*R. S. Phillips* (Los Angeles, Calif.).

**Edwards, R. E. On functions whose translates are independent.** Ann. Inst. Fourier Grenoble **3** (1951), 31–72 (1952).

The paper is concerned with translates of functions defined on a group. Let $G$ be a locally compact abelian group and denote the character group by $\hat{G}$. For $E=L_2(G)$, $f\varepsilon E$, and $A$ a subset of $G$, the author defines $I(f,A)$ to be the closed subspace spanned by the translates $[f(x+a)]$ of $f$ where $a$ ranges over $A$. A function $f\varepsilon E$ has its translates independent if, whenever $A$ is a closed subset of $G$ and $a\varepsilon G-A$, then $f(x+a)$ does not belong to $I(f,A)$. If $G$ is discrete, then $f$ has its translates independent if and only if $[F(\chi)]^{-1}\varepsilon L_2(G)$ where $F(\chi)=\int_G \chi(x)f(x)dx$ is the Fourier transform of $f$. For $G$, the real numbers $R$ or the real numbers mod $2\pi$, the author obtains sufficient conditions in order that a function have its translates independent. It is interesting to note that for $G$ the real numbers mod $2\pi$, the Fourier coefficients of such a function may vanish on a sufficiently sparse set of integers. These results are then extended to $R^m$. Also considered is the case $E$ equal to continuous functions on $G$, the topology being uniform convergence in every compact subset. The paper closes with a

discussion of the connection of this problem with the theory of normed rings and with various possible extensions.

*R. S. Phillips* (Los Angeles, Calif.).

**Bose, S. K. On generalised Laplace integral.** Ganita **2**, 33–43 (1951).

In this sequel to earlier papers [Bull. Calcutta Math. Soc. **41**, 9–27, 59–67, 68–76, 221–222 (1949); **42**, 43–48 (1950); J. Indian Math. Soc. (N.S.) **14**, 29–34 (1950); Ganita **1**, 16–22 (1950); these Rev. **11**, 28, 173, 174; **12**, 95, 256; **13**, 458] the author obtains two generalizations of a result on Laplace transforms, and evaluates several "Whittaker transforms". [Reviewer's remark: of the two general results formulated in section I, the second concerns a finite sum of Laplace transforms and is a trivial "generalization". The first, and more general, result depends on the behavior of a certain unidentified function, and there is no evidence to show that the function exists or has the required behavior.]

*A. Erdélyi* (Pasadena, Calif.).

**Kac, A. M. On computation of the quadratic criterion of the quality of regulation.** Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 362–364 (1952). (Russian)

Let $P(p)/Q(p)$ be the Laplace transform of $f(t)$, where $Q(p)=a_0p^n+\cdots+a_n$ and $P(p)$ is a polynomial of degree $\leqq n-1$; let $P(p)P(-p)=g_0p^{2n-2}+\cdots+g_{n-2}p^2+g_{n-1}$. The author proves that $J=\int_0^\infty[f(t)]^2dt=(-1)^{n-1}G/(2a_0D)$, where

$$D=\begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ 0 & a_0 & a_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdot & a_{n-2} & a_n \end{vmatrix}$$

and $G$ is obtained by replacing the first row of $D$ by the row $g_0, \cdots, g_{n-1}$. A convenient method for the numerical computation of $G$ is also given. *J. L. Massera* (Montevideo).

**Delerue, Paul. Sur quelques images en calcul symbolique à trois ou $n$ variables.** Bull. Sci. Math. (2) **76**, 119–128 (1952).

The author derives several (formal) rules for Heaviside calculus in 3 or more variables, and uses these rules to evaluate some images. The examples involve Bessel functions, exponential integral functions, and related functions. He indicates briefly the application of multiple operational calculus to the solution of integral equations of the convolution type in several variables. *A. Erdélyi*.

## *Special Functions*

**Höfinger, E. Zur Theorie der hypergeometrischen Funktionen.** Monatsh. Math. **56**, 126–136 (1952).

The author proves that if $\delta\equiv xd/dx$,

$$[P(\delta)-xQ(\delta)]f(x)=[R(\delta)-xS(\delta)]g(x)=0,$$

where $P, \cdots, S$ are polynomials with constant coefficients, then

$$[P(\delta)S(1-\delta)-xQ(\delta)R(1-\delta)]\int_C f(xt)g(t)dt=0$$

provided that $C$ is equivalent to a contour closed on the Riemann surface of the integrand. He uses this result to establish certain integrals and integral representations regarding hypergeometric functions. *A. Erdélyi*.

**Toscano, Letterio. Formule di trasformazione e sviluppi sulle funzioni ipergeometriche a due variabili.** Ann. Mat. Pura Appl. (4) **33**, 119–134 (1952).

The principal results of this paper are two groups of formulas. The first group lists nine cases of special values of parameters and arguments which reduce Appell's hypergeometric series of two variables to a (generalized) hypergeometric series in one variable. Not all these reduction formulas are claimed to be new. The second group of formulas gives sixteen expansions of Appell's series in a series of Appell's series, or a series of Gauss' series (or products of such series). [The author appears to be unaware of a paper by Burchnall and Chaundy [Quart. J. Math., Oxford Ser. **11**, 249–270 (1940); these Rev. **2**, 287] which gives an elegant derivation of most of the formulas of the second group; the remainder of this second group being simple particular cases of results established by Burchnall and Chaundy.] *A. Erdélyi* (Pasadena, Calif.).

**Halphen, Etienne. Sur une famille de fonctions.** C. R. Acad. Sci. Paris **235**, 684–686 (1952).

The functions introduced in this paper are well-known confluent hypergeometric functions, in essence parabolic cylinder functions. *A. Erdélyi* (Pasadena, Calif.).

**Slater, L. J. Integrals representing general hypergeometric transformations.** Quart. J. Math., Oxford Ser. (2) **3**, 206–216 (1952).

D. B. Sears' general transformations of hypergeometric series [Proc. London Math. Soc. (2) **53**, 138–157 (1951); these Rev. **13**, 33] are here proved by means of Barnes' integrals. [This is essentially C. S. Meijer's proof [Nederl. Akad. Wetensch., Proc. **49**, 227–237 (1946), Theorem E; these Rev. **8**, 156].] Certain transformations of basic hypergeometric series are proved similarly. [See also the following review.] *A. Erdélyi* (Pasadena, Calif.).

**Slater, L. J. An integral of hypergeometric type.** Proc. Cambridge Philos. Soc. **48**, 578–582 (1952).

Let $t>0$, $q=e^{-t}$, and use the abbreviation

$$(a+\sigma;k)=\prod_{i=1}^{k}(1-q^{a_i+\sigma}).$$

The author considers the integral

$$I=\frac{1}{2\pi i}\int_{-i\pi/t}^{i\pi/t}\prod_{n=0}^{\infty}\frac{(a+s+n;k)\ (b-s+n;L)}{(c+s+n;M)\ (d-s+n;N)}ds.$$

Evaluating this integral in two different ways, a master formula is obtained from which all known transformations of basic hypergeometric series can be deduced. [$I$ is the basic analogue of Meijer's $G$-function.] *A. Erdélyi*.

**Karamata, J. Sur certains développements asymptotiques avec application aux polynomes de Legendre.** Acad. Serbe Sci. Publ. Inst. Math. **4**, 69–88 (1952).

Let

$$G_n(\theta)=\sum_{r=0}^{\infty}C_r(n)e^{r\theta i} \tag{1}$$

be a trigonometric series whose coefficients, $C_r(n)$, possess, for large $n$, an asymptotic expansion of the form

$$G_r(n)=\sum_{s=0}^{k}\frac{\gamma_s(r)}{q_s(n)}+O\left(\frac{1}{q_k(n)}\right)\quad \text{as } n\to\infty, \tag{2}$$

where the functions $q_s(n)$, $s=0, 1, 2, \cdots$, are of increasing orders of infinity. On substitution of (2) in (1), a formal asymptotic expansion of $G_n(\theta)$, for large $n$, is obtained. The author proves the validity of this asymptotic expansion under the assumptions that (1) and the $(k+1)$ trigonometric expansions $\sum_{r=0}^{\infty}\gamma_s(r)e^{r\theta i}$ are Abel-summable, that $C_r(n)=A_rB_r(n)$ where $\{A_r\}$ is a completely monotonic sequence $\{B_r(n)\}$ for fixed $n$, a sequence whose $K$th difference $\downarrow 0$ as $r\uparrow\infty$, and that $\gamma_s(r)$ is a polynomial in $r$ whose degree is $<K$. As an application of the general results, the author derives two asymptotic expansions of $P_n(\cos\theta)$ for $0<\epsilon\leqq\theta\leqq\pi-\epsilon$, and $n\to\infty$, and also two companion expansions for Legendre functions of the second kind.

*A. Erdélyi* (Pasadena, Calif.).

**Aljančić, S. Beitrag zur Theorie der Gegenbauerschen Polynome.** Srpska Akad. Nauka. Zbornik Radova 18, Matematički Inst. 2, 113–128 (1952). (Serbo-Croatian. German summary)

Gegenbauer polynomials $C_n^{\nu}(x)$ are the ultraspherical polynomials [Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939, section 4.7; these Rev. 1, 14]. The author uses two trigonometric expansions [Szegö, op. cit., (4.9.22) and (4.9.24)] in conjunction with Karamata's method [see preceding review] and an inequality of Fejér's to obtain the asymptotic representation as $n\to\infty$ [Szegö, Theorem 8.211.11] of, and an inequality [Szegö, Theorem 7.33.2, gives a more precise version] for Gegenbauer polynomials.

*A. Erdélyi* (Pasadena, Calif.).

**Hohlov, R. V. On an asymptotic expression for the associated Laguerre functions.** Doklady Akad. Nauk SSSR (N.S.) 85, 975–976 (1952). (Russian)

From an integral representing $L_n^{\nu}(x)$, an asymptotic formula is obtained in terms of the Airy function and its derivative. No estimate of the order of the error is given but it would seem that the formula is designed for the case that $n$ is large in comparison with $x$, and $\nu$ is near the critical value $2(nx)^{1/2}$.

*A. Erdélyi* (Pasadena, Calif.).

**Takahashi, Takehito. Generalized spherical harmonics as representation matrix elements of rotation group.** J. Phys. Soc. Japan 7, 307–312 (1952).

Generalized spherical harmonics are connected with certain Jacobi polynomials. The transformation properties of these functions, for a rotation of the unit sphere, are derived here by means of the rotation group. [Reviewer's remark: The paper is not very clearly written. It is known that the effect of a rotation on spherical harmonics in three dimensions is expressed by a matrix of four-dimensional surface harmonics whose variables are the Cayley parameters of the rotation. These seem to be the author's generalized harmonics, and their transformation is the matrix relation expressing the group property of rotations in three-dimensional space.]

*A. Erdélyi* (Pasadena, Calif.).

**Conolly, B. W. An application of the "Faltung" formula.** Ganita 2, 50–52 (1951).

The author uses the product theorem of Laplace integrals to evaluate three integrals. Of these, the first is a finite integral of a product of Bessel functions, the second has a product of hypergeometric series, and the third a product of Whittaker's functions.

*A. Erdélyi* (Pasadena, Calif.).

**Markovitz, Hershel. A property of Bessel functions and its application to the theory of two rheometers.** J. Appl. Phys. 23, 1070–1077 (1952).

The mathematical part gives recurrence relations for the coefficients of the expansion of $J_m(x)Y_m(y)-Y_m(x)J_m(y)$ in powers of $x$ and $(y-x)/x$, and in case $m=0$ or 1, formulas for some coefficients and for some infinite series associated with them.

*A. Erdélyi* (Pasadena, Calif.).

**Agostinelli, Cataldo. Sulle funzioni epicicloidali e loro applicazione ad alcuni problemi di fisica matematica.** Ann. Mat. Pura Appl. (4) 33, 165–210 (1952).

The author gives a detailed account of the results presented in two previous notes [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 316–320 (1950); 11, 339–344 (1951); these Rev. 11, 440; 13, 843], with some applications to boundary-value problems.

*A. Erdélyi.*

**MacDonald, William M., III, Richardson, John M., and Rosenberry, Leon P. Representation of nonlinear field functions by Thiele semi-invariants.** Quart. Appl. Math. 10, 284–289 (1952).

Let $\partial T(x)/\partial x$ be a function such that its Fourier transform is $\phi(t)=\int_{x_1}^{x_2}e^{itx}(\partial T/\partial x)dx$ and such that its cumulants (Thiele semi-invariants) $\kappa_n$, defined by the identity $\phi(t)\equiv\exp\left(\sum_n(it)^n\kappa_n/n!\right)$, exist. Let

$$\mathfrak{J}(x)=T(x)-\tfrac{1}{2}[T(x_1)+T(x_2)]$$

and $\mu_n=\int_{-\infty}^{\infty}x^n\mathfrak{J}^s(\partial\mathfrak{J}/\partial x)dx$. Let

$$P=(\kappa_0, \kappa_1, \kappa_2, 0, 0, \cdots, 0, \cdots)$$

and let $\mu_n$ be regarded as a function of $\{\kappa_n\}$. The authors then write

$$\mu_n=(\mu_n)_P+\sum_{i=3}^{\infty}\left(\frac{\partial\mu_n}{\partial\kappa_i}\right)_P\kappa_i+\tfrac{1}{2}\sum_{i=3}^{\infty}\sum_{j=3}^{\infty}\left(\frac{\partial^2\mu_n}{\partial\kappa_i}\partial\kappa_j\right)_P\kappa_i\kappa_j+\cdots$$

and present formulas for $(\mu_n)_P$, $(\partial\mu_n/\partial\kappa_i)_P$, $(\partial^2\mu_n/\partial\kappa_i\partial\kappa_j)_P$. In a similar manner the case in which $\mathfrak{J}$ represents an $n$-tuple of functions is then treated. The authors state that the $\mu_n$ have been tabulated up to $n=5$ for $s=1$ and $s=2$ while only the first terms have been tabulated up to $n=4$ for $s=3$.

The reviewer noticed the following errors. In equation (1.1) $(\phi t)$ should be $\phi(t)$; (2.2) contains a surplus $dt$; in (2.3) $x_1$ should be $x$ (this results in some difficulty in the subsequent displayed assertion). In the displayed expression following (2.4) the $\lambda$'s should be omitted; in (3.6) each $\phi$ should be $\phi_s$; on the line following (4.2) $\Psi(t)$ should be $\Psi_i(t)$; in equation (3.3) $\prod_{r=0}^{s}$ should be replaced by $\prod_{r=1}^{s}e^{-ia t_0}$; in (3.4) $-s-1$ should be $-s$; in (3.9) $\tau_s$ should be $t_s$. Some simplification could be introduced by the observation that $\phi(\sum_i\xi_i)=\prod_i\phi(\xi_i)$ so that the multiple integrals (e.g., in (3.4) and in the third equation after (3.11)) could be replaced by a single (simpler) integral, raised to the exponent $s$.

*H. D. Block* (Ames, Iowa).

## *Harmonic Functions, Potential Theory*

**Jackson, Lloyd. The principle of the maximum for generalized subharmonic functions.** Portugaliae Math. 11, 69–74 (1952).

Subharmonicity is here generalized by replacing the family of dominating harmonic functions by an abstract class $\{F\}$ of functions, which (like harmonic functions) are

determined uniquely by their boundary values. With continuous sub-$\{F\}$ functions so defined and continuous super-$\{F\}$ functions defined similarly, the following form of the maximum principle is obtained: if $g$ and $h$ are, respectively, continuous sub-$\{F\}$ and super-$\{F\}$ functions on a domain $D$, and $D'$ is a closed subdomain of $D$ such that $g \leqq h$ on the boundary of $D'$, then $g \leqq h$ on $D'$. Further analogues of properties of subharmonic functions are established, and a number of applications are given, based on the fact that solutions of certain elliptical partial differential equations satisfy the postulates on $\{F\}$. In particular, the author derives in this way a Phragmén-Lindelöf theorem for schlicht functions in a horizontal half-strip.

*M. G. Arsove* (Seattle, Wash.).

**Komatu, Yûsaku. Eine gemischte Randwertaufgabe für einen Kreis.** Proc. Japan Acad. 28, 339–341 (1952).

Let $U(\varphi)$ and $V(\varphi)$ be integrable on the arcs $a<\varphi<b$ and $b<\varphi<a+2\pi$, respectively, of the unit circle, and let $U(\varphi)$ satisfy the further condition that $U(\varphi)\{(\varphi-a)(b-\varphi)\}^{-1/2}$ be integrable on $a<\varphi<b$. The author gives an explicit expression for a function $u(z)$, harmonic in $|z|<1$, which tends to $U(\varphi)$ for almost all $\varphi$ in $a<\varphi<b$, and whose normal derivative tends to $V(\varphi)$ for almost all $\varphi$ in $b<\varphi<a+2\pi$, the approach being non-tangential. The result is related to a more general boundary-value problem solved by A. Signorini [Ann. Mat. Pura Appl. (3) 25, 253–273 (1916)].

*A. J. Lohwater* (Ann Arbor, Mich.).

**Komatu, Yûsaku. On functions harmonic in a circle, with special reference to Poisson representation.** Proc. Japan Acad. 28, 342–346 (1952).

Four theorems are proved concerning harmonic functions with Poisson representation, the most general of these being the following: A function $u(z)$, harmonic in $|z|<1$ and tending almost everywhere to a given integrable boundary function $f(\theta)$, has a Poisson integral representation in terms of $f(\theta)$ if and only if (1) $u(z)$ is of bounded mean modulus in $|z|<1$ and (2) the singular part of the mass distribution occurring in the Poisson-Stieltjes representation of $u(z)$ as a result of (1) vanishes identically. *A. J. Lohwater.*

**Finzi, Arrigo. Sur les conditions de validité du théorème de Liouville.** Bull. Sci. Math. (2) 76, 110–112 (1952).

The author remarks that the usual proofs of Liouville's theorem on the existence of conformal transformations of spaces of dimension higher than two require second order differentiability of the equations of transformation. Although he cannot show that Liouville's theorem can be proved assuming only first order differentiability of the equations, he offers a geometrical lemma, which he hopes may be useful in this direction, to the effect that seven spheres which are externally tangent in a certain way are mutually tangent internally to an eighth sphere. In this connection, the author seems unaware of a proof of Liouville's theorem given by A. Capelli [Ann. Mat. Pura Appl. (2) 14, 227–237 (1886)]. *A. J. Lohwater.*

**Siegel, Keeve M. Three-dimensional conformal transformations.** J. Aeronaut. Sci. 19, 281–282 (1952).

The author observes that the Kelvin inversion transformation of potential theory cannot be used to solve three-dimensional boundary value problems in hydrodynamics because the boundary conditions are not of the right type, and that hence conformal mapping is not a useful technique in three-dimensional problems. *D. Gilbarg.*

**Šilov, G. E. Vectorially smooth functions.** Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 176–184 (1951).

L'A. généralise les notions de divergence et de rotationnel d'un champ plan de vecteurs au cas où les composantes du champ peuvent n'être pas dérivables. Les définitions qu'il utilise ont été introduites depuis longtemps (cf. par exemple, la thèse de N. Théodoresco [Paris, 1931]), mais l'A. ne cite pas ses devanciers. Quelques applications qu'il tire de cette idée ancienne, paraissent, à première vue, originales. Citons: (1) Le critère pour qu'une famille de fonctions vectorielles, régulières au sens de l'A. soit complète dans un domaine. (2) L'étude du gradient d'un potentiel de masses à densité continue. *J. Kravtchenko* (Grenoble).

**Müller, Claus. Über die Grundoperationen der Vektoranalysis.** Math. Ann. 124, 427–449 (1952).

In $E_3$, let $G$ denote a regular region [Kellogg, Potential theory, Springer, Berlin, 1929, p. 113], $\|G\|$ its volume, $F$ its boundary, $N$ the exterior normal to $F$, and $da$ the element of area on $F$. If $P \varepsilon G$ and $G_n \subset G$, define $G_n \to P$ to mean that $G_n$ lies in a small sphere with center at $P$ if $n$ is sufficiently large. [Although it is not explicitly demanded in this definition (p. 429), some proofs (cf. (1.8)) seem to require $P \varepsilon G_n$, and in the introduction the regions $G_n$ "sich auf den Punkt $P$ zusammenziehen."] If the expressions

$$\frac{1}{\|G_n\|}\int_{F_n} N_n \cdot V da_n, \quad \frac{1}{\|G_n\|}\int_{F_n} N_n \times V da_n$$

have limits as $G_n \to P$ that are independent of the sequence $G_n$, these limits are denoted respectively by div $V$ and rot $V$. If the scalar $\phi \varepsilon C'$, the Laplacian is defined as $\Delta\phi = \operatorname{div} \operatorname{grad} \phi$. [The similar definition of grad $\phi$ with integrand $\phi N_n$ is not used. These definitions appear in several texts on vector analysis and theoretical physics. In his lectures Gibbs used the above definition of div and a definition of rot suggested by Stokes' Theorem [Gibbs and Wilson, Vector analysis, 5th ed., Yale Univ. Press, 1925, pp. 187, 194]. No discussion known to the reviewer has used these definitions in a consequential way; the familiar expressions involving derivatives of the components, valid if $V \varepsilon C'$, have ordinarily been introduced at once and thenceforth used exclusively.] An example is given of a continuous $V$ for which div $V$ exists and is continuous although $V$ is nondifferentiable on a dense set. The familiar integral identities hold provided the relevant div's and rot's are continuous, and if $\Delta\phi = 0$ in $G$, then $\phi$ can be expressed in the familiar way in terms of $\phi$ and grad $\phi$ on $F$.

It has been shown by Wintner [Amer. J. Math. 72, 731–738 (1950); these Rev. 12, 704] that there exists a scalar function $\rho$ continuous on $E_3$ whose logarithmic potential $\phi$ fails to have $\phi_{xx}$. It is shown here that if $\rho$ is continuous (not necessarily Hölder continuous) in $G \subset E_3$ and $\phi$ is its Newtonian potential, then $\Delta\phi$ exists at all points of $G$ and $\Delta\phi = -4\pi\rho$. Maxwell's equations are treated for fields periodic in time that are generated by electric and magnetic current densities with continuous divergences. Finally, the equations $\Delta\phi + \rho\phi = 0$ and $\Delta\phi + \lambda\rho^2\varphi = 0$ are discussed with $\rho$ continuous. *F. A. Ficken* (Knoxville, Tenn.).

**Holmberg, B. A remark on the uniqueness of the potential determined from the asymptotic phase.** Nuovo Cimento (9) 9, 597–604 (1952).

It is shown that the impossibility of determining the potential between two particles from the asymptotic phases when there exist also bound states, comes from the fact that

it is not possible to extend the orthogonality relations given by Froberg and Hylleraas to the discrete eigenfunctions. An expression in terms of these functions is given for the difference between two phase-equivalent potentials with the same energy spectrum. Finally, (in the case when only one bound state exists) an explicit formula is given for constructing this difference if one special potential and its corresponding eigenfunction is known. (From the author's summary.) *N. Levinson* (Cambridge, Mass.).

**Teissier du Cros, François. Sur le lien entre les notions "champ réel autonome" et "cellule d'harmonicité".** C. R. Acad. Sci. Paris **235**, 600–601 (1952).

If a function $f(x_1, \cdots, x_n)$ in a domain $D$ of the real Euclidean $E_n$ is harmonic in $D$, then it is real-analytic there, and beyond that, in the neighborhood of every point its power series, if arranged as one series of homogeneous polynomials, is absolutely convergent in the largest sphere in $D$ whose center is at that point. The author makes some remarks on the sizes of the class of all harmonic functions within the larger class of all analytic functions having the property stated. *S. Bochner* (Princeton, N. J.).

**Payne, L. E., and Weinstein, Alexander. Capacity, virtual mass, and generalized symmetrization.** Pacific J. Math. 2, 633–641 (1952).

In a meridian plane, the authors consider a region which generates a body of revolution $B[n]$ in $n$-dimensional space, and they associate with this body a volume $V[n]$, capacity $C[n]$, and virtual mass $M[n]$. They apply a symmetrization $S_q$ to $B[n]$ which can be interpreted as a Schwarz symmetrization of the corresponding body $B[q+1]$. For $0<q\leqq n-1$, $V[n]$ and $C[n]$ do not increase under $S_q$, and for $n-1\leqq q\leqq n+1$, $M[n]$ does not increase under $S_q$. The proof applies for fractional values of $n$ and $q$ and is based on an inequality due to Weinberger [Proc. Nat. Acad. Sci. U. S. A. **38**, 611–613 (1952); these Rev. **14**, 24]. The result generalizes a remark of the reviewer and D. C. Spencer [J. Rational Mech. Anal. **1**, 359–409 (1952); these Rev. **14**, 102]. *P. R. Garabedian* (Stanford, Calif.).

## *Differential Equations*

**Bojanić, Ranko. Ein Existenzsatz über die Lösungen einer Klasse impliziter Differentialgleichungen erster Ordnung.** Srpska Akad. Nauka. Zbornik Radova **18**, Matematički Inst. **2**, 137–142 (1952). (Serbo-Croatian. German summary)

F. A. Valentine's proof [Univ. California Publ. Math. (N.S.) **2**, 77–84 (1944); these Rev. **5**, 239] for the existence and uniqueness of a solution to the problem $x'-f(t,x,x')=0$, $x(t_0)=x_0$, is slightly modified by the author. *M. Golomb* (Lafayette, Ind.).

**Viktorovskiĭ, E. E. On a general existence theorem for solutions of differential equations connected with the consideration of integral inequalities.** Mat. Sbornik N.S. **31**(73), 27–33 (1952). (Russian)

Let (1) $y'=f(x,y)$, where $f$ is continuous in $y \in (-\infty,+\infty)$ for almost all $x \in [x_0, x_0+\alpha]$, measurable in $x$ for each $y$ and $|f|\leqq M(x)\varphi(|y|)$, where $M$ is summable, $\varphi$ continuous, positive, increasing and such that $\int_0^\infty dy/\varphi(y)=\infty$. Then, given any $y_0$, there are two absolutely continuous solutions $u_0(x)$, $v_0(x)$ of (1) which are the infimum and supremum of all absolutely continuous solutions satisfying $y(x_0)=y_0$. *J. L. Massera* (Montevideo).

**Kamke, E. Über den Existenzbereich der Integrale der quasilinearen Differentialgleichung 1. Ordnung.** Acad. Serbe Sci. Publ. Inst. Math. **4**, 61–68 (1952).

Let $i, k, r=1, \cdots, n$, let $v$ denote $(y_i)$, and let a subscript $x$, $y_r$, or $z$ denote a derivative. In the slab $|x-\xi|\leqq a$, all $v$ and $z$, let $f_k(x,v,z)$ and $g(x,v,z)$ have $f_{ky_r}$, $f_{kz}$, $g_{y_r}$, and $g_z$ continuous and absolutely $\leqq A$, and let $\omega(v)$ have $\omega_{y_r}$ continuous and absolutely $\leqq C$. Then $z_x+\sum f_k z_{y_k}=g$ has a unique solution in the slab $|x-\xi|\leqq\alpha$, all $v$, with $z(\xi,v)=\omega(v)$. Perausówna [Ann. Soc. Polon. Math. **12**, 1–5 (1934)] and Ważewski [ibid. **12**, 6–15 (1934)] found best values for $\alpha$: $A^{-1}(C+1)^{-1}$ for $n=1$, and

$$A^{-1}(n-1)^{-1}\log[n(C+1)(nC+1)^{-1}]$$

for $n\geqq 2$. Their proofs are improved here by exploiting the characteristic functions of the system $y_k'=f_k(x,v)$. *F. A. Ficken* (Knoxville, Tenn.).

**Germay, R. H. Sur l'intégration des équations récurro-différentielles par la méthode des approximations successives.** Bull. Soc. Roy. Sci. Liège **21**, 260–265 (1952).

The author considers a system of differential equations of the form

$$\frac{dy_n}{dx}=F_n(x, y_n, y_{n+1}) \quad (n=1, 2, 3, \cdots),$$

where the typical function $F_n$ is not given explicitly, but is defined as the limit of a sequence of functions $F_{n1}$, $F_{n2}$, $F_{n3}$, $\cdots$. Making appropriate assumptions concerning the boundedness and continuity of the functions $F_{np}$, and using a modification of Picard's method of successive approximations, he proves the existence of a solution of the system of equations, satisfying a set of initial conditions of the form $y_n(x_0)=y_0$ $(n=1, 2, 3, \cdots)$. *L. A. MacColl.*

**Cafiero, Federico. Sul fenomeno di Peano nelle equazioni differenziali ordinarie del primo ordine.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 51–61 (1951).

Let $S$ denote the strip $a\leqq x\leqq b$, $|y|<\infty$, let $f(x,y)$ be continuous and bounded on $S$, and let $P \in S$ belong to $E_r$ (or $E_l$) if from $P$ there issues a unique solution of the equation $y'=f(x,y)$ extending to the right (or left) of $P$. Theorem: If $E_r$ (or $E_l$) is everywhere dense in $S$, then the complement of $E_r\cdot E_l$ has measure zero. Application to families of such equations, e.g., $y'=f(x,y)+\lambda$. Extension with $f$ measurable in $x$. [Cf. the following review.] *F. A. Ficken* (Knoxville, Tenn.).

**Cafiero, Federico. Sul fenomeno di Peano nelle equazioni differenziali ordinarie del primo ordine. II.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 123–126 (1951).

Let $f(x,y,\lambda,\gamma)$ be continuous in the cylinder $a\leqq x\leqq b$, $c\leqq y\leqq d$, $|\lambda|<\infty$, $|\gamma|<\infty$, and suppose that $f$ is monotonic in $\gamma$; for almost all $(\lambda,\gamma)$ the equation $y'=f(x,y,\lambda,\gamma)$ then has a unique solution through almost every $(x,y)$. [Cf. the preceding review.] *F. A. Ficken* (Knoxville, Tenn.).

**Cafiero, Federico. Sulla classe delle equazioni differenziali ordinarie del primo ordine, i cui punti di Peano costituiscono un insieme di misura lebesguiana nulla.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 127–137 (1951).

Let $\Sigma$ denote the set of functions $f(x,y)$ continuous on a finite rectangle $R$. Let $\Sigma_1$ denote the real line and, with $k$

and $(\bar{x}, \bar{y}) \in R$ fixed, let $\Sigma_1$ denote the subset of those $f \in \Sigma$ with $f(\bar{x}, \bar{y}) = k$; then $\Sigma = \Sigma_1 \times \Sigma_2$. With suitable measures $\mu$ in $\Sigma_1$ and $m$ in $\Sigma_2$, define a measure $(\mu m)$ in $\Sigma$. Theorem: Unless $f$ belongs to a subset of $\Sigma$ with $(\mu m)$-measure zero, there is a unique solution of $y' = f(x, y)$ through almost every point of $R$. [Cf. the two preceding reviews.]

*F. A. Ficken* (Knoxville, Tenn.).

**Valiron, Georges. Fonctions entières et équations différentielles.** Bull. Sci. Math. (2) 76, 144–148 (1952).

Let $P_j = P_j(z)$, $Q_j = Q_j(z)$ be polynomials and consider the differential equation

$$(1)\quad W^{(n)} + P_1 W^{(n-1)} + \cdots + P_{n-1} W' = Q_0 + Q_1 W + \cdots + Q_k W^k.$$

Anastassiadis [same Bull. (2) 76, 57–64 (1952); these Rev. 13, 943] proved that equation (1) admits no solutions which are entire functions of mean type if $k > 1$. Valiron points out, using known results, that this statement can be strengthened to read as follows: Equation (1) admits no entire solutions of finite order if $k > 1$. The following theorem is also given. Let $G(\lambda_1, \cdots, \lambda_{p+1})$ be a polynomial and let $\Phi(z)$ be a function holomorphic about the point at infinity and having infinity as an essential point. Then the differential equation $G(z, y, y', \cdots, y^{(p)}) = \Phi(y)$ cannot have a solution of the form $y = z^{\alpha + i\beta} F(z)$, where $F(z)$ is holomorphic about the point at infinity and $|z|^{\alpha} |F(z)|$ is unbounded there.

*F. G. Dressel* (Durham, N. C.).

**Morris, J. On the solution of linear simultaneous differential equations with constant coefficients by a process of isolation.** Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2623** (11, 420), 7 pp. (1952).

The author illustrates his procedure first for the initial value problem $(D - m_1)(D - m_2)x = f(t)$, $x = x_0$, and $Dx = \dot{x}_0$ when $t = 0$, where $D = d/dt$ and $m_1$ and $m_2$ are complex constants. His method is similar to that of solving for $(D - m_2)x$ and then for $(D - m_1)x$ and determining the constant of integration in each step from the initial conditions. The function $x(t)$ is found by eliminating $Dx$ from those two expressions. He then extends this procedure to corresponding problems involving equations of higher order, when the roots $m_i$ of the characteristic equations are known.

*R. V. Churchill* (Ann Arbor, Mich.).

**Krasovskiĭ, N. N. Theorems on stability of motions determined by a system of two equations.** Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 547–554 (1952). (Russian)

The author considers the stability of the trivial solution $x = 0$, $y = 0$ of $dx/dt = f_1(x) + ay$, $dy/dt = f_2(x) + by$, where $f_1/x$, $f_2/x$ satisfy certain boundedness conditions. The method used is an extension of the Liapounoff technique which consists of forming an appropriate quadratic form in $x$ and $y$ and investigating its dependence upon $t$. Using similar techniques, the equation of Liénard,

$$d^2x/dt^2 + f(x)dx/dt + g(x) = 0$$

is investigated. *R. Bellman* (Santa Monica, Calif.).

**Vinograd, R. È. On a criterion of instability in the sense of Lyapunov of the solutions of a linear system of ordinary differential equations.** Doklady Akad. Nauk SSSR (N.S.) 84, 201–204 (1952). (Russian)

Consider the system $dy/dt = A(t)y$ where $A(t)$ is a real continuous or piecewise continuous matrix. Let $B(t) = \frac{1}{2}(A + A^*)$, where $A^*$ is the transpose of $A$. Suppose there exists a differentiable orthogonal matrix $U(t)$ such that $U^{-1}BU = D(t)$ is a diagonal matrix. If $y = U(t)x$, then clearly $x' = [D(t) + C(t)]x$ where $C(t)$ is skew-symmetric. Criteria for instability are given in terms of the characteristic roots of $B(t)$ and the magnitude of $C(t)$. Non-linear cases are also considered.

*N. Levinson* (Cambridge, Mass.).

**Erugin, N. P. Theorems on instability.** Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 355–361 (1952). (Russian)

The instability of the solution $x = 0$ of systems of the form $\dot{x} = f(t, x)$, $x$ being an $n$-vector, is considered by the author. Several general criteria are derived using methods similar to the second method of Lyapunov. *J. L. Massera.*

**Erugin, N. P. On a problem of the theory of stability of systems of automatic regulation.** Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 620–628 (1952). (Russian)

The author considers the problem of the boundedness of the solutions of the system $dx/dt = f(x) + ay$, $dy/dt = bx + cy$, on the basis of certain assumptions concerning $f(x)$.

*R. Bellman* (Santa Monica, Calif.).

**Barbašin, E. A. On the stability of solution of a nonlinear equation of the third order.** Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 629–632 (1952). (Russian)

Study of the stability of the solution $x = 0$ of the third order equation

$$(1)\quad \dddot{x} + a\ddot{x} + \varphi(\dot{x}) + f(x) = 0$$

where $a$ is a positive constant, $f$ is continuously differentiable for all $x$, $\varphi(y)$ is continuous for all $y$, and $f(0) = \varphi(0) = 0$. Set:

$$F(x) = \int_0^x f(x)dy, \quad \Phi = \int_0^y \varphi(y)dy,$$

$$w(x, y) = aF(x) + f(x)y + \Phi(y).$$

Replace also (1) by

$$(2)\quad \dot{x} = y, \quad \dot{y} = z - ay, \quad \dot{z} = -f(x) - \varphi(y).$$

It is proved that the origin as a solution is asymptotically stable whatever the initial position under one of the following two sets of conditions: I. $f(x)/x > 0$ for $x \neq 0$; $\varphi(y)/y - f'(x) > 0$ for $y \neq 0$; $w(x, y) \to +\infty$ with $(x^2 + y^2)^{1/2}$. II. There exist two positive numbers $h_1$, $h_2$ such that $ah_2 - h_1 > 0$ and such that $f(x)/x > h_1$ for $x \neq 0$;

$$a\varphi(y)/y - f'(x) > ah_2 - h_1.$$

In the special case $f(x) = bx$, $b > 0$, the same stability result is obtained under the conditions:

$$\frac{\varphi(y)}{y} > \frac{b}{a} \text{ for } y \neq 0; \quad \Phi(y) - b\frac{y^2}{2a} \to +\infty \text{ as } |y| \to \infty.$$

[References: Malkin, same journal 16, 365–368 (1952); these Rev. 14, 48; Barbašin, Učenye Zapiski Moskov. Gos. Univ. 135, Matematika 2, 110–133 (1948); these Rev. 11, 443]. *S. Lefschetz* (Princeton, N. J.).

**Kunin, I. A. Determination of a finite region of initial deviations for which the motions remain asymptotically stable for a system of two equations of the first order** Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 539–546 (1952). (Russian)

Discussion of the optimum elliptic region of the nature stated in the title for a system

$$\dot{x} = p_{11}x + p_{12}y + P'(x, y) = P(x, y)$$
$$\dot{y} = p_{21}x + p_{22}y + Q'(x, y) = Q(x, y)$$

where: (a) the $p_{ij}$ are constants such that

$$p_{11}+p_{22}<0; \quad p_{11}p_{22}-p_{12}p_{21}>0$$

(stability conditions of Hurwitz); (b) $P'$ and $Q'$ are holomorphic in the whole plane and begin with terms of degree at least two. The treatment is based on the utilization of a positive definite Lyapunov function

$$V=Ax^2+2Bxy+Cy^2$$

whose coefficients are determined by the condition

$$\dot{V}'=\frac{\partial V}{\partial x}(p_{11}x+p_{12}y)+\frac{\partial V}{\partial y}(p_{21}x+p_{22}y)=-\alpha x^2-\beta y^2$$

where $\alpha$, $\beta$ are positive. If one sets

$$\dot{V}=P\frac{\partial V}{\partial x}+Q\frac{\partial V}{\partial y}=\dot{V}'+P'\frac{\partial V}{\partial x}+Q'\frac{\partial V}{\partial y},$$

then the origin is interior to a region where $\dot{V}<0$. If one constructs the curve $\dot{V}=0$ and inscribes in it the maximum ellipse $V=\text{const.}$, then that ellipse bounds a region $\Omega(\alpha,\beta)$ of asymptotically stable initial positions. It is clear that one may write $V=\alpha V_1+\beta V_2=\alpha(V_1+\gamma V_2)$, $\gamma>0$, where $V_1$, $V_2$ are two functions $V$ whose expression is readily found. The point of the contact of the optimum ellipse $V=\text{const.}$ with $\dot{V}=0$ has for locus $d(\dot{V}_2/\dot{V}_1)/dt=0$ and on that locus one must take the optimum point. Some geometric considerations are given to guide the choice. *S. Lefschetz.*

**Turrittin, H. L. Asymptotic expansions of solutions of systems of ordinary linear differential equations containing a parameter.** Contributions to the Theory of Nonlinear Oscillations, vol. II, pp. 81–116. Princeton University Press, Princeton, 1952.

In the system of differential equations

$$\text{(1)} \qquad \epsilon^h\dot{X}=A(t,\epsilon)X$$

$h$ is a non-negative integer, $\epsilon$ is a small complex parameter, $X$ is a column vector of $N$ components, and the square matrix $A(t,\epsilon)$ has an asymptotic expansion $A(t,\epsilon)\sim\sum_{k=0}^{\infty}\epsilon^kA_k(t)$, valid for $t$ in an interval $D$: $a\leqq t\leqq b$, and for $\epsilon$ in a region $R$ containing the origin as a boundary point. The elements of $A(t,\epsilon)$ are analytic functions of $\epsilon$ inside $R$. The author shows first that the system (1) possesses $N$ independent formal vector solutions of the form

$$\text{(2)} \qquad X_i=\left[\sum_{k=0}^{\infty}\epsilon^{k/r}X_{ik}(t)\right]e^{q_i(t,\epsilon)} \quad (i=1,\cdots,N)$$

where $r$ is a certain positive integer, and

$$q_i(t,\epsilon)=\int_a^t\epsilon^{-h}\rho_i(t,\epsilon)dt,$$

with $\rho_i(t,\epsilon)=\sum_{k=0}^{hr-1}\epsilon^{k/r}\rho_{ik}(t)$. The proof consists in a successive simplification of (1) by means of a sequence of suitable changes of the dependent variable. This requires an investigation of the canonical form of certain matrices related to $A(t,\epsilon)$. The interval $a\leqq t\leqq b$ must therefore be taken so small that the multiplicities of the eigenvalues and the elementary divisors of these matrices do not change there.

Next it is shown that if the formal series (2) are truncated, they represent asymptotically $N$ actual solutions of (1) to a finite number of terms. The analysis resembles that of G. D. Birkhoff and R. E. Langer [Proc. Amer. Acad. Arts Sci. 58, 49–128 (1923)]. In the proof the regions $D$ and $R$ are assumed to be chosen so that those functions $\rho_i(t,\epsilon)$ which are not identically equal are distinct there. The actual solutions so obtained depend on the number of terms in the truncated series. In order to obtain solutions that are asymptotically equal to the infinite series (2), a method due to Tamarkin and Besikowitsch [Math. Z. 21, 119–125 (1924)] is applied to a system obtained from (1) by a suitable linear transformation. The proof proceeds by induction using solutions already obtained for a reduction of the order of the system. An application to the non-homogeneous system corresponding to (1) is included.

The results, but not the methods, of this paper are similar to those of Hukuhara [Mem. Fac. Eng. Kyushu Imp. Univ. 8, 249–280 (1937)]. Hukuhara does not treat the nonhomogeneous equation. *W. Wasow* (Los Angeles, Calif.).

**Dragilev, A. V. Periodic solutions of the differential equation of nonlinear oscillations.** Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 85–88 (1952). (Russian)

The equation $\ddot{x}+f(x,\dot{x})\dot{x}+g(x)=0$, and the corresponding system (1) $\dot{x}=v$, $\dot{v}=-f(x,v)v-g(x)$ are considered ($\dot{}=d/dt$). It is shown that a theorem due to Levinson and Smith [Duke Math. J. 9, 382–403 (1942); these Rev. 4, 42] which asserts the existence of a periodic solution of (1) under certain conditions can be strengthened slightly. Let (2) $\dot{x}=v$, $\dot{v}=-f^*(x,v)v-g(x)$ be another system and suppose $f$ and $f^*$ are continuous, except perhaps on the $x$-axis, and satisfy Lipschitz conditions in every bounded domain not containing points of the $x$-axis; $f(x,v)<0$ in a neighborhood of the origin; $g$ is continuous and satisfies a Lipschitz condition in every finite interval, $xg(x)>0$ for $|x|>0$; $f(x,v)\geqq f^*(x,v)$. Then if (2) possesses a periodic solution, so does (1). *E. A. Coddington* (Los Angeles, Calif.).

**Tihonov, A. On boundary conditions containing derivatives of order higher than the order of the equation.** Amer. Math. Soc. Translation no. 76, 28 pp. (1952).

Translated from Mat. Sbornik N.S. 26(68), 35–56 (1950); these Rev. 11, 440.

**Uno, Toshio. On the curves defined by some differential equations.** Math. Japonicae 2, 119–126 (1952).

The author considers a differential equation of the form

$$\frac{dy}{dx}=\frac{AY+BX}{CY+DX},$$

where $X$, $Y$ are real-valued analytic functions of the real variables $x$, $y$, and where $A$, $B$, $C$, $D$ are real parameters such that $AD-BC>0$. As one parameter is varied, one obtains a family of differential equations; and this family is said to be capable of being ordered if a trajectory defined by one equation and a trajectory defined by another equation are never tangent, except possibly at one or more of the fixed singular points. The author is concerned with the behavior of the family of trajectories under variations of the parameters, especially in the cases in which the families of equations can be ordered. In particular, he discusses the appearance and disappearance of limit cycles, and the effects on the family of trajectories of a focus changing from the stable to the unstable type. Most of the conclusions seem to be based mainly upon intuition, and some of the latter parts of the paper are wanting in clarity. *L. A. MacColl.*

**Uno, Toshio, and Yokomi, Rieko. On some mode of appearance of limit cycles.** Math. Japonicae 2, 117–118 (1952).

The authors consider the differential equation

$$\frac{dy}{dx}=\frac{f(x,y,\lambda)}{g(x,y,\lambda)},$$

where $f$ and $g$ are polynomials in $x$, $y$, with coefficients which are analytic functions of $\lambda$. They are concerned with the displacement, confluence, and disappearance of singularities caused by variations of $\lambda$. It is shown that in one rather special situation the confluence and subsequent disappearance of a node and a saddle point is associated with the appearance of a new limit cycle. *L. A. MacColl.*

**Nordon, Jean. Nouveaux cas d'intégrabilité par quadratures d'une équation différentielle remarquable du premier ordre. II.** C. R. Acad. Sci. Paris **235**, 1181–1182 (1952).

[For part I see same C. R. **232**, 140–141 (1951); these Rev. **12**, 501.] The author determines certain functions $f(x)$ such that the equation $dy/dx=[f(x)-y^2]^{1/2}$ can be solved by quadratures. *L. A. MacColl* (New York, N. Y.).

**Terracini, A. Aspetti proiettivi nella teoria delle equazioni differenziali.** Ann. Univ. Ferrara. Parte I. **8** (1948-50), 183–195 (1951).

Geometric definitions of some types of differential equations having a projective or topological character are examined, and their constructive content and the possibility of a geometric representation of the coefficients are analysed. *Author's summary.*

**Volpato, Mario. Un criterio di unicità per un problema ai limiti relativo all'equazione $y''=f(x, y, y', \lambda)$.** Ann. Univ. Ferrara. Parte I. **8** (1948–50), 33–40 (1951).

The author gives a sufficient condition under which the characteristic value problem $y''(x)=f[x, y(x), y'(x), \lambda]$ $y(a)=\alpha$, $y(x_0)=\gamma$, $y(b)=\beta$ $(a<x_0<b)$ possesses a solution for one, and only one, value of $\lambda$. *L. A. MacColl.*

**Pucci, Carlo. Formule di maggiorazione per un integrale di una equazione differenziale lineare del secondo ordine.** Ann. Mat. Pura Appl. (4) **33**, 49–90 (1952) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **336** (1952).

Several bounds are found for the solution of an equation

$$[\Theta(x)y'(x)]'+A(x)y(x)=f(x)$$

satisfying the boundary conditions $\alpha_1 y(a)-\alpha y'(a)=\gamma$, $\beta_1 y(b)+\beta y'(b)=\gamma_1$, which generalize previous results by Picone and Krylov. *J. L. Massera* (Montevideo).

**Avakumović, Vojislav G. Über die Randwertaufgabe zweiter Ordnung.** Acad. Serbe Sci. Publ. Inst. Math. **4**, 1–8 (1952).

This is essentially the same as an earlier paper [Srpska Akad. Nauka. Zbornik Radova **7**, Mat. Inst. **1**, 1–16 (1951); these Rev. **13**, 237], although no reference is made to the earlier paper. The hypothesis $|f(x, y)|\leqq M$ is replaced by the slightly more general one $|f(x, y)|\leqq h(x)$, which necessitates a change in the definition of the region $D$.

*M. Golomb* (Lafayette, Ind.).

**Bojanić, Ranko. Über die Konvergenz einer Folge von Polynomen.** Srpska Akad. Nauka. Zbornik Radova **18**, Matematički Inst. **2**, 133–136 (1952). (Serbo-Croatian. German summary)

If $p_1(x)=x(1-x)$, $p''_{n+1}(x)+\alpha p_n(x)=0$, $p_n(0)=p_n(1)=0$, $n=1, 2, \cdots$, then $\sum_{n=1}^{\infty} p_n(\frac{1}{2})=2\alpha^{-1}(\sec\frac{1}{2}\alpha^{1/2}-1)$ for $|\alpha|<\pi^2$. This result, used in the preceding paper by V. G. Avakumović, is proved by the author in a more elementary fashion. *M. Golomb* (Lafayette, Ind.).

**Mohr, Ernst Die Konstruktion der Greenschen Funktion im erweiterten Sinne** J. Reine Angew. Math. **189**, 129–140 (1951).

If $L$ is the linear differential operator defined by

$$L(u)=\sum_{\nu=1}^{n} a_\nu(x)u^{(\nu)}(x),$$

the Green's function $g(x, \xi)$ of the differential equation $L(u)=0$ for a given set of homogeneous boundary conditions is defined as a solution of $L(u)=0$ which satisfies the boundary conditions and has $n-2$ continuous derivatives in the given interval, but whose $(n-1)$th derivative has a given jump at a point $\xi$. It is well known that, in those cases in which the homogeneous differential system has a solution with $n-1$ continuous derivatives, this Green's function does not exist and is replaced by a generalized Green's function $g_1(x, \xi)$ which does the same service as $g(x, \xi)$ in the solution of the associated inhomogeneous system. $g_1(x, \xi)$ satisfies the same homogeneous boundary conditions but is a solution of the homogeneous equation $L(g_1)=p(x)$, where the function $p(x)$ is subject to certain conditions. Continuing earlier work by W. W. Elliott [Amer. J. Math. **50**, 243–258 (1928)], the author gives a characterization of the most general function $p(x)$ which may be used for this purpose and he investigates various normalizations to which $p(x)$ may be made subject. *Z. Nehari* (St. Louis, Mo.).

**Volpato, Mario. Sulle condizioni sufficienti per l'unicità degli integrali di una equazione differenziale alle derivate parziali del primo ordine.** Ann. Univ. Ferrara. Parte I. **8** (1948–50), 137–149 (1951).

A uniqueness theorem of Ważewski [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **18**, 372–376 (1933)] is extended so as to apply to more problems and to yield uniqueness in larger regions of more general shape.

*F. A. Ficken* (Knoxville, Tenn.).

**Cinquini, Silvio. Un teorema di unicità (in forma generalizzata) per l'equazione $p=f(x, y, z, q)$.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 255–259 (1951).

Announcement of extension of uniqueness results in an earlier paper [Ann. Mat. Pura Appl. (4) **32**, 121–155 (1951); these Rev. **13**, 845]. *F. A. Ficken* (Knoxville, Tenn.).

**Bajada, Emilio. L'equazione $p=f(x, y, z, q)$ e l'unicità.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 163–167 (1952).

A function $z(x, y)$ defined by the system $z_x=f(x, y, z, z_y)$, $z(0, y)=0$ is unique in an isosceles right triangle with hypotenuse on $x=0$ provided (i) $f$ satisfies a Lipschitz condition in $z$ and in $q$; (ii) $z$ is continuous in $x$, $y$; (iii) $z$ is absolutely continuous in $x$ for fixed $y$; (iv) $z$ has a total differential except at a set with projection on the $x$-axis of measure zero; and (v) $|z_x|$ is dominated by an integrable function of $x$ alone. *J. M. Thomas* (Durham, N. C.).

**Gallissot, François. Transformations infinitésimales et intégration des équations différentielles de la mécanique.** C. R. Acad. Sci. Paris **235**, 1277–1278 (1952).

Let $V_{2n+1}$ be a manifold of dimension $2n+1$, $\Omega$ an exterior quadratic differential form of rank $2n$, and $\Sigma$ the system of differential equations associated to $\Omega$. Suppose $d\Omega=0$. Then to every infinitesimal transformation under which $\Omega$ is invariant corresponds a first integral of $\Sigma$, and vice versa. If $X^1, \cdots, X^n$ are $n$ infinitesimal transformations under

which $\Omega$ is invariant and are two-by-two in involution, then $\Sigma$ can be integrated by quadratures and there are $n$ closed Pfaffian forms $\pi^1, \cdots, \pi^n$, so that $\Omega = \sum_{\alpha=1}^{n} \pi^\alpha \wedge \pi_\alpha$. The results have applications to analytical dynamics.

*S. Chern* (Chicago, Ill.).

**Mönnig, Paul. Über die Lösung der Hamilton-Jacobischen Differentialgleichung durch Trennung der Variablen.** Math. Z. 56, 49–56 (1952).

Nach einem Satz von Jacobi ist eine Funktion $F(\mathfrak{x}, \mathfrak{y})$ dann und nur dann ein Integral der kanonischen Gleichungen

$$\frac{dx_i}{dt} = H_{p_i}, \quad \frac{dp_i}{dt} = H_{x_i}, \quad i = 1, 2, \cdots, n;$$

$$\mathfrak{x} = \{x_1, \cdots, x_n\}; \quad \mathfrak{y} = \{p_1, \cdots, p_n\}$$

wenn sie eine Lösung der linearen partiellen Differentialgleichung

$$(H, F) = \sum_{i=1}^{n} (H_{p_i} F_{x_i} - H_{x_i} F_{p_i}) = 0$$

ist. Verfasser untersucht den Fall, in welchem Funktionen $F^i$ existieren, die nicht nur die Poissonschen Klammer $(H, F)$ sondern sogar deren einzelne Summanden zum Verschwinden bringen. Das so entstehende System wird zunächst durch Klammerbildungen ergänzt und auf die Jacobische Gestalt gebracht. Die Integrale des in Rede stehenden Systems verwendet nunmehr Verfasser zur Ermittlung einer vollständigen Lösung der Hamilton-Jacobischen Differentialgleichung

$$H(\mathfrak{x}, \mathfrak{y}) = k, \quad \mathfrak{y} = \text{grad } W.$$

Die Methode führt auf eine Verallgemeinerung des bereits von Stäckel angegebenen Verfahrens der Integration durch Separation der Variablen. Diese Trennung der Variablen hängt von dem Bestehen eines von Levi-Cività angegebenen partiellen Bedingungssystem zweiter Ordnung ab. Verfasser gewinnt notwendige und hinreichende Bedingungen für die Erfüllbarkeit dieser von Levi-Cività angegebenen Bedingungsgleichungen und behandelt weiterhin einen Spezialfall. Durch geeignete Spezialisierung gewisser Funktionen dieses Sonderfalles gelingt es auch die Resultate Dall'Aqua's wieder zu gewinnen, die dieser in dem für die Mechanik wichtigen Falle erhalten hatte.

*M. Pinl* (Dacca).

**Rachajsky, M. B. Application des fonctions caractéristiques dans la théorie géométrique des caractéristiques.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 536–546 (1952).

Considering the first order differential equation

$$F(x, y, z, p, q) = 0, \quad F_p \neq 0,$$

necessary and sufficient conditions are discussed which must be satisfied to construct geometrically complete integrals $Z = f(x, y, C_1, C_2)$, eliminating $C_3$ from the relations

$$y = \varphi(x, C_1, C_2, C_3) \quad \text{and} \quad z = \psi(x, C_1, C_2, C_3).$$

Similarly, are discussed the differential equation

$$F(x_1, \cdots, x_n, z, p_1, \cdots, p_n) = 0, \quad F_{p_1} \neq 0,$$

and the system in involution

$$F_k(x_1, \cdots, x_n, z, p_1, \cdots, p_n) = 0,$$

$$\Delta = \frac{\partial(F_1, \cdots, F_m)}{\partial(p_1, \cdots, p_m)} \neq 0, \quad [F_k, F_s] = 0, \quad k, s = 1, 2, \cdots, m < n.$$

*M. Pinl* (Dacca).

**Halilov, Z. I. Cauchy's problem for an operator equation in partial derivatives.** Doklady Akad. Nauk SSSR (N.S.) 85, 959–962 (1952). (Russian)

The equation is of the type (1) $\partial u/\partial t = \sum A_\lambda(t) D_\lambda u + f$, where the sum is finite, $D_\lambda$ a derivation with respect to certain variables $x_1, \cdots, x_n$, $u = u(t, x)$ and $f = f(t, x)$ are elements of a Banach space $B$ defined for all $x$ and $0 \leqq t \leqq T$, and $A_\lambda(t)$ are linear operators on $B$ depending on $t$. By means of a Fourier transformation, Cauchy's problem for (1) is reduced to the problem of finding an operator $V = V(t, \tau, \alpha)$ satisfying (2) $dV/dt = \sum A_\lambda(t)\lambda(i\alpha)V$ and $V(\tau, \tau, \alpha) = 1$ ($\lambda(i\alpha) = (i\alpha_1)^{k_1} \cdots$ if $D_\lambda = (\partial/\partial x_1)^{k_1} \cdots$; $\alpha_1, \cdots$ real). A boundedness condition for (2) then gives a necessary and sufficient condition that Cauchy's problem for (1) be correctly set. When $B$ is finite-dimensional, the theory is due to Petrowsky [Bull. Univ. d'Etat Moscou. Sect. A. 1, no. 7 (1938)].

*L. Gårding* (Lund).

**Myškis, A. D. On the transition from the usual first boundary problem to the modified one.** Mat. Sbornik N.S. 31(73), 128–135 (1952). (Russian)

L'A. cherche à affranchir la technique de résolution du problème classique de Dirichlet, posé relativement à l'équation elliptique:

$$\text{(E)} \qquad \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x_1, \cdots, x_n)\frac{\partial f}{\partial x_j}\right] = 0, \quad a_{ij} = a_{ji},$$

des restrictions que l'on impose, dans les exposés traditionnels, à la frontière $K$ (constituée pour simplifier, d'un nombre fini de composantes connexes) du domaine $G$ où l'on se propose de construire l'inconnue $f$ connaissant ses valeurs $\varphi$ sur $K$ ($\varphi$ est supposée continue). A cet effet, l'A. considère dans $G$ la classe $A$, non vide, de fonctions $f$, $n$ fois différentiables dans $G$ et non constantes sur les surfaces $S$ [à $(n-1)$ dimensions] le long desquelles $\partial f/\partial n > 0$: dans toute la suite nous entendrons par $S$ la frontière assez régulière de tout le domaine $H \varepsilon S$, à normale intérieure $n$.

A chaque (E), l'A. associe une fonctionnelle linéaire réelle $\Gamma(f, S)$ telle que $\Gamma(f, S) > 0$ pour tout $f$ non-$\varepsilon A$ assujettie aux conditions: $\partial f/\partial n > 0$ sur $S$ et $f = $ cte sur $S$. Le problème de Dirichlet est alors remplacé par le problème M ci-après: trouver $f \varepsilon A$ prenant sur $K$ des valeurs frontières $\psi$ ($\varphi - \psi$ étant constante sur chaque composante connexe de $K$) et telle que $\Gamma(f, S) = 0$ pour toute surface $S \varepsilon G$. L'A. montre successivement: 1) La difference de deux solutions du problème M est une constante; 2) le problème M admet une solution au moins; 3) si le problème de Dirichlet admet une solution correspondant à $\varphi$, le problème M associé admet aussi une solution.

Enfin, l'A. donne quelques propriétés de stabilité de la solution du problème M et précise la manière dont celle-ci dépend de $K$ et de $\varphi$ (théorèmes de convergence).

*J. Kravtchenko* (Grenoble).

**Èĭdus, D. M. On the solution of boundary problems by the method of finite differences.** Doklady Akad. Nauk SSSR (N.S.) 83, 191–194 (1952). (Russian)

The Dirichlet boundary-value problem has been dealt with by the method of finite differences by L. A. Lyusternik [Mat. Sbornik 33, 173–201 (1926)]; I. G. Petrovskiĭ [Uspehi Matem. Nauk 8, 161–170 (1941); these Rev. 3, 123]; R. Courant, K. Friedrichs, and H. Lewy [Math. Ann. 100, 32–74 (1928)]. In the present paper, other boundary value problems for the elliptic equation

$$Lu = \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) = f(x), \quad a_{ij} = a_{ji},$$

are treated by the method of finite differences. Particular attention is given to the boundary-value problem where the boundary condition is

$$Pu = \sum_{i,j=1}^{m} a_{ij}\frac{\partial u}{\partial x_i}\cos(\nu, x_j) = 0.$$

It is remarked that a similar procedure is applicable to mixed boundary problems and eigenvalue problems for the same operator $L$. *J. B. Diaz* (College Park, Md.).

**Èĭdus, D. M. On continuous dependence of characteristic functions on the region.** Doklady Akad. Nauk SSSR (N.S.) **83**, 365–367 (1952). (Russian)

Consider the eigenvalue problem: (*) $L(u)+\lambda u=0$ in $\Omega$, $u=0$ on $\Gamma$, where $\Omega$ is a bounded open set in $(x_1, \cdots, x_m)$-space, $\Gamma$ is the boundary of $\Omega$, and

$$L(u) = \sum_{i,j=1}^{m} \partial(a_{ij}\partial u/\partial x_j)/\partial x_i,$$

with $a_{ij}=a_{ji}$, the functions $a_{ij}$ being thrice continuously differentiable on $\Omega+\Gamma$. Let

$$E(\varphi, \psi) = \int_\Omega \sum_{i,j=1}^{m} a_{ij}(\partial\varphi/\partial x_i)(\partial\psi/\partial x_j)d\Omega;$$

$E(\varphi)=E(\varphi, \varphi)$; $H(\varphi, \psi)=\int_\Omega \varphi\psi d\Omega$; $H(\varphi)=H(\varphi, \varphi)$, and

$$D(\varphi) = \int_\Omega \sum_{i=1}^{m}(\partial\varphi/\partial x_i)^2 d\Omega.$$

Let $D$ be the class of once continuously differentiable functions defined in $\Omega$, each of which vanishes in a boundary strip of $\Omega$, and $\mathring{D}$ be the completion of the class $D$ with respect to the norm defined by $\|\varphi\|^2=H(\varphi)+D(\varphi)$. It has been shown previously in the paper reviewed above that the eigenvalue problem (*) possesses a countable sequence of eigenfunctions $u_k$ in $\mathring{D}$, with corresponding eigenvalues $\lambda_k$, and that each $u_k$ minimizes $E(v)$ over $\mathring{D}$, subject to the conditions $H(v)=1$, $H(v, u_i)=0$, $i=1, \cdots, k-1$. Let $\Omega_n$ be a sequence of open sets which exhausts $\Omega$, and consider the corresponding eigenvalue problem $(*)_n$ for $\Omega_n$, it being supposed that the functions of $\mathring{D}_n$ are defined to be zero on $\Omega-\Omega_n$, so that $\mathring{D}_n\subset\mathring{D}$. Call the corresponding eigenvalues and eigenfunctions $u_{kn}$ and $\lambda_{kn}$. It was shown in Courant and Hilbert, Methoden der Mathematischen Physik, Bd. I [Springer, Berlin, 1931], that as $n\rightarrow\infty$, $\lambda_{kn}\rightarrow\lambda_k$. It is shown here that, in the sense of the metric $\|\ \|$ above, the distance between $u_{kn}$ and the subspace of eigenfunctions of $\lambda_k$ approaches zero as $n\rightarrow\infty$. A similar question is considered for the eigenvalue problem analogous to (*), with boundary condition $\sum_{i,j}^{m} a_{ij}(\partial u/\partial x_i)\cos(\nu, x_j)=0$ on $\Gamma$.

*J. B. Diaz* (College Park, Md.).

**Germain, Paul, et Bader, Roger. Application de la solution fondamentale à certains problèmes relatifs à l'équation de Tricomi.** C. R. Acad. Sci. Paris **231**, 1203–1205 (1950).

Unter Verwendung von Bezeichnungen und Resultaten einer vorhergehenden Note [Germain, dieselben C. R. **231**, 1116–1118 (1950); diese Rev. **14**, 177] benutzen die Verfasser eine Fundamentallösung der Differentialgleichung von Tricomi in einer Umgebung des Punktes $P$ ($\vartheta=0, \sigma=0$), die in $P$ selbst unendlich wird wie $2\pi\sigma_0^{-1/2}\log\overline{MP}$. Diese Lösung besteht aus zwei in $P$ und $M$ symmetrischen Ausdrücken, aus welchen die Verfasser auch die Greensche Funktion des Problems gewinnen. Damit gelingt auch die Lösung der in einer früheren Arbeit [ibid. **231**, 268–270 (1950); diese Rev. **14**, 177] aufgeworfenen Cauchyschen Probleme für Tricomi's Differentialgleichung—das alles zunächst nur für den Fall, dass $P$ im elliptischen Gebiet liegt. Beim Übergang ins hyperbolische Gebiet erhalten die Verfasser das folgende Resultat: der Wert einer Lösung $\psi(P)$ der Tricomischen Differentialgleichung kann durch Anwendung der Greenschen Formel für beliebige Lage des Punktes $P$ im hyperbolischen Gebiet $(D)$ durch die Randwerte von $\psi$ und dessen erste Ableitungen auf dem Rand von $(D)$ ausgedrückt werden. Als spezielle Anwendung auf das Problem von Tricomi wird insbesondere noch der Fall untersucht, dass die Werte von $\psi$ auf einem Bogen $\Gamma$ der elliptischen Halbebene in bestimmter Weise vorgegeben werden. *M. Pinl* (Dacca).

**Cole, Julian D. Note on the fundamental solution of $wy_{vv}+y_{ww}=0$.** Z. Angew. Math. Physik **3**, 286–297 (1952).

The differential equation

$$(*)\qquad wy_{vv}+y_{ww}=0$$

occurring in the gas-dynamical theory of transonic flow is of mixed type, elliptic for $w>0$, hyperbolic for $w<0$. The basic problem is to obtain a representation of the fundamental singular solution of (*) with a singularity in the elliptic half-plane $w>0$. A fundamental solution of (*), $T(w, v-v_1; w_1)$, is defined as a singular solution of (*) such that

$$y(w, v) = \int\int_{-\infty}^{+\infty} T(w, v-v_1; w_1)Q(w_1, v_1)dw_1dv_1$$

is a solution of $wy_{vv}+y_{ww}=-Q(w, v)$. Solutions of such a kind can be continued into the hyperbolic region $w<0$. A supplementary condition must be introduced which is familiar from the theory of purely hyperbolic equations. For hyperbolic problems the direction of propagation of disturbances along the characteristics must be prescribed in addition to the differential equations. Then regions of influence and domains of dependence can be specified. For the equation (*) of mixed type a similar requirement on the behavior of the solution in the hyperbolic region makes the fundamental solution $T$ unique. This method allows one to consider the limiting case $w_1=0$ using some transformations of hypergeometric functions. Finally the logarithmic singularity for $w_1>0$ is discussed. *M. Pinl* (Dacca).

**Ladyženskaya, O. On integrals of hyperbolic equations.** Doklady Akad. Nauk SSSR (N.S.) **79**, 925–927 (1951). (Russian)

The author considers solutions of the hyperbolic equation

$$(*)\qquad u_{tt} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - a(x)u$$

subject to the boundary conditions $(\alpha u+\beta\partial u/\partial N)_S=0$ where $S$ is a smooth closed surface in $n$-dimensional space bounding a domain $\Omega$ and $\alpha$ and $\beta$ are prescribed functions. Certain integrals $\int_\Omega \omega_k(u, v)d\Omega+\int_S \omega_{k-1}(u, v)dS$, where $\omega_k(u, v)$ is a bilinear expression involving derivatives to the $k$th order of $u$ and $v$ and where $u$ and $v$ are solutions of (*) satisfying the boundary conditions, are discussed. A collection of such integrals is called complete if estimates for a certain integral $H_k(u)$ involving squares of the derivatives (to the $k$th order) of $u$ can be obtained in terms of this collection. The problem of finding complete systems of positive integrals for arbitrary order $k$ is solved. *M. H. Protter.*

**Ladyženskaya, O. A. On the solution of a mixed problem for hyperbolic equations.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 545–562 (1951). (Russian)

Proofs of results previously announced [Doklady Akad. Nauk SSSR (N.S.) **73**, 647–650 (1950); these Rev. **12**, 338].

*M. H. Protter* (Princeton, N. J.).

**Sauer, R. Dreidimensionale Probleme der Charakteristikentheorie partieller Differentialgleichungen.** Z. Angew. Math. Mech. **30**, 347–356 (1950). (German. English, French, and Russian summaries)

The author discusses the method of characteristics in three-dimensional problems with particular reference to gas dynamics. In the first part he treats problems which are essentially two-dimensional because of certain symmetries (e.g., axial symmetry) and then problems with approximate symmetry whose solutions can be considered linear perturbations of the symmetrical ones and can therefore also be treated by the two-dimensional method of characteristics. A discussion of these cases can also be found in earlier work of the author [e.g., Einführung in die theoretische Gasdynamik, 2nd ed., Springer, Berlin, 1951; these Rev. **14**, 107]. In the second part the author considers problems which are properly three-dimensional and, after a general introduction to the theory of characteristics for hyperbolic differential equations in three independent variables, outlines the procedure in numerical solution of initial value problems by a method of finite differences analogous to that used in two-dimensional problems. *D. Gilbarg.*

**Atkinson, F. V. On a theorem of K. Yosida.** Proc. Japan Acad. **28**, 327–329 (1952).

The theorem referred to is given in the same Proc. **27**, 214–215 (1951) [these Rev. **13**, 656; in this review read $\alpha<m^{1/2}2^{-1/2}$ instead of $\alpha<m^{1/2}g^{-1/2}$]. The author shows that Yosida's condition $h(x)=O(\exp\alpha|x|)$ may be replaced by $h(x)=O(|x|^{-(n-1)/2}\exp(|x|m^{1/2}))$ and that this condition is in a certain sense the best possible. *W. Feller.*

**García, Godofredo. On the integration and properties and the integral curves of the complete diffusion equation.** Actas Acad. Ci. Lima **15**, 3–24 (1952). (Spanish)

The complete diffusion equation referred to is

$$D\cdot\Delta_2 u-\operatorname{div}(\mathbf{v}u)+F(u)=u_t$$

where $D$ is constant, $u$ a concentration and $0<u<1$. The author first considers the slightly different case where $\mathbf{v}\equiv 0$ and $F(u)$ is replaced by $(a+b\epsilon(t))u$, with $a$ and $b$ constant. He surveys various methods of integration. The complete equation is considered mainly in one dimension with $\mathbf{v}\equiv 0$. Various forms of $F(u)$ are suggested, some of which ultimately lead to an ordinary differential equation which can be integrated. *W. Feller* (Stockholm).

**Sestini, Giorgio. Esistenza di una soluzione in problemi analoghi a quello di Stefan.** Rivista Mat. Univ. Parma **3**, 3–23 (1952).

Stefan initiated the study of heat conduction across the shifting boundary between two phases of a substance—change of phase being accompanied by absorption or emission of heat. The author studies the one-dimensional problem of an infinite solid, bounded by $x=0$ and $x=a$. Heat is generated at $x=0$, while $x=a$ is insulated. The boundary between phases is $x=\alpha(t)$. The temperature (with the critical temperature as the origin) is $u^{(1)}(x,t)$ for $0<x<\alpha(t)$, $u^{(2)}(x,t)$ for $\alpha(t)<x<a$. The differential systems to be studied are:

$$(\mathrm{A}_1)\quad\begin{cases} k_1\dfrac{\partial^2u^{(1)}}{\partial x^2}=\dfrac{\partial u^{(1)}}{\partial t}, & 0<x<\alpha(t),\\[2mm] h_1\left[\dfrac{\partial u^{(1)}}{\partial x}\right]_{x=0}=-H(t), & \\[2mm] u^{(1)}[\alpha(t),t]=0, & \alpha(0)=0;\end{cases}$$

$$(\mathrm{A}_2)\quad\begin{cases} k_2\dfrac{\partial^2u^{(2)}}{\partial x^2}=\dfrac{\partial u^{(2)}}{\partial t}, & \alpha(t)<x<a,\\[2mm] u^{(2)}(x,0)=f(x), & \\ u^{(2)}[\alpha(t),t]=0, & \alpha(0)=0,\\[2mm] \left[\dfrac{\partial u^{(2)}}{\partial x}\right]_{x=a}=0. & \end{cases}$$

Here $u^{(1)}$ and $u^{(2)}$ are linked by the equation

$$\left[h_2\frac{\partial u^{(2)}}{\partial x}-h_1\frac{\partial u^{(1)}}{\partial x}\right]_{x=\alpha(t)}=\rho_2L\frac{d\alpha}{dt}.$$

$L$ is the latent heat of fusion, $h_i$ the coefficient of thermal conductivity, $k_i$ the coefficient of thermal diffusivity, $\rho_i$ the density (assumed constant for each phase), $H(t)$ the rate of heat delivered to the substance at $x=0$.

Hypotheses: As to $f(x)$, the initial temperature, $f(0)=0$, $f'(a)=0$, $f'(x)\leqq 0$, $f''(x)\geqq 0$. Also, $H(t)$ is monotonically non-decreasing (though the results are later extended to the case in which $H(t)$ is a function of bounded variation), continuous with a continuous first derivative.

$\alpha(t)$, $u^{(1)}$, $u^{(2)}$ are found by successive approximations. With $\alpha_0(t)=(\rho_2L)^{-1}\int_0^tH(\tau)d\tau$, the author finds $u_{n-1}^{(i)}(x,t)$ from $\alpha_{n-1}(t)$ by systems similar to $\mathrm{A}_1$ and $\mathrm{A}_2$, $\alpha_n$ from $u_{n-1}^{(1)}$ and $u_{n-1}^{(2)}$ by integration.

This extends recent work by Rubinstein, Evans, and others, particularly Evans [Quart. Appl. Math. **9**, 185–193 (1951); these Rev. **13**, 243]. The time-interval over which Sestini's method guarantees existence and uniqueness of a solution is four times that of Evans.

To add in the comprehension of the article we add the text of an extremal theorem (referred to as T. E.) which, although essential to the argument at many points, is not stated. Its source is Picone's "Appunti d'analisi superiore" [Rondinella, Naples, 1940, pp. 710–712; these Rev. **3**, 144]. Let

$$(1)\quad E(u)\equiv\sum_{h,k}^{1,2}a_{hk}(X)\frac{\partial^2u}{\partial x_h\partial x_k}+\sum_{h=1}^{2}b_h(X)\frac{\partial u}{\partial x_h}+c(X)u=f(X)$$

be an equation of elliptic-parabolic type in a region $A$: (a) If $c(X)<0$ throughout $A$, a solution of (1) cannot have a negative minimum [positive maximum] there if $f(x)\leqq 0$ [$f(x)\geqq 0$]. (b) Let $A$ be bounded and let it be possible to define on $A+FA$ a function $\omega(X)$, regular with respect to $E(u)$, positive, and such that $E(\omega)<0$ in $A$; then if, in $A$, $f(x)\leqq 0$ [$\geqq 0$], every solution of (1) regular in $A+FA$, non-negative [non-positive] on $FA$, remains such on $A$. The existence of the function $\omega(X)$ is assured by the hypotheses: $a_{11}>0$; $c\leqq 0$; $b_1/a_{11}$ has an upper bound.

*E. S. Allen* (Ames, Iowa).

**Paterson, Stewart. On certain types of solution of the equation of heat conduction.** Proc. Glasgow Math. Assoc. **1**, 48–52 (1952).

For each of the partial differential equations $f_t=f_{rr}$, $f_t=f_{rr}+f_r/r$, and $f_t=f_{rr}+2f_r/r$ particular solutions are found under the assumption $f=f(\zeta)$ where $\zeta=\phi(r)\cdot\psi(t)$.

*F. G. Dressel* (Durham, N. C.).

**Minasyan, R. S. Unsteady heat flow in a prismatic body with transverse section in the form of a right angle in the presence of unsteady heat sources.** Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 533–538 (1952). (Russian)

The planar non-stationary heat distribution is considered in a prismatic body whose transverse cross-section has the form of a carpenter's square. The initial heat distribution within the region is given. The temperature along the boundary of the region is a given function of time. It is assumed that there exist unsteady heat sources within the region, and that the temperature $U(x, y, t)$ is symmetrically distributed, i.e., $U(x, y, t)=U(y, x, t)$. The method of Laplace transforms is used to reduce the problem to the solution of an infinite system of linear algebraic equations, which is shown to possess a solution. *H. P. Thielman.*

**Bergman, Stefan. The coefficient problem in the theory of linear partial differential equations.** Trans. Amer. Math. Soc. 73, 1–34 (1952).

The author continues to apply his integral operator method [Mat. Sbornik N.S. 2(44), 1169–1198 (1937); Duke Math. J. 13, 419–458 (1946); these Rev. 8, 274] to the investigation of the solutions of linear partial differential equations. The operators in question transform analytic functions in one or two complex variables into solutions of given partial differential equations in two and three variables, respectively. These operators are of sufficient formal simplicity to permit the extension of many known results on functions of one or two complex variables to the corresponding classes of solutions of partial differential equations. In the present paper this technique is applied with a view to establishing connections between certain local series expansions of such functions and a number of properties "in the large". *Z. Nehari* (St. Louis, Mo.).

**Neamtan, S. M., and Vogt, E. Boundary conditions in the mechanics of fields.** Canadian J. Physics 30, 684–698 (1952).

The authors discuss variational principles for scalar and vector fields involving Lagrangeans which contain second order derivatives of the field functions. The variations of the field quantities allowed are those which satisfy prescribed conditions on a surface bounding a region of three-dimensional Euclidean space. The class of allowed boundary conditions is found to include the "linear homogeneous condition", namely, the boundary condition which states that the normal derivative of a field quantity is proportional to that quantity at a point of the boundary. Also included in this class are the boundary conditions which state that the field quantities or their normal derivatives vanish on the boundary. *A. H. Taub* (Urbana, Ill.).

**Davis, R. B. A boundary value problem for third-order linear partial differential equations of composite type.** Proc. Amer. Math. Soc. 3, 751–756 (1952).

With $b$'s depending on $x$ and $y$, consider

$$(*)\qquad b_1u_{xxx}+b_2u_{xxy}+b_3u_{xyy}+b_4u_{yyy}+\cdots=0$$

where the dots denote terms involving no derivatives of order $>2$. If $\sum_0^3 b_{k+1}p^{3-k}q^k$ has just one simple real zero $p/q$, then (*) is called composite at $(x, y)$. Let $R$ be a bounded simply connected region, let the $b$'s belong to $C'$ in $R$, and let (*) be composite at every point of $R$. It is shown that then new coordinates $(z, t)$ can be introduced in $R$ so that (*) becomes $Lu\equiv(u_{zz}+u_{tt})_z+\cdots=0$. Let $T$ be a closed subregion of $R$ with a rectifiable boundary $W$, and let $W'$ be a curve in $T$ from $W$ to $W$ cutting every $z$-curve ($t=$const.) in $T$ exactly once without being tangent to any. With $L$ linear, with no term in the second derivative $u_{tt}$, and under suitable assumptions (including estimates for the derivatives of the Green's function for Laplace's operator and $T$ that are known to hold if $W$ is continuously curved), it is shown that either the homogeneous problem $Lu=0$, $u=0$ on $W'$, $u_z=0$ on $W$ has a nontrivial solution, or the nonhomogeneous problem $Lu=f$, $u=g$ on $W'$, $u_z=h$ on $W$ has a unique solution. The method is generally similar to that of E. E. Levi [see Courant and Hilbert, Methoden der mathematischen Physik, vol. II, Springer, Berlin, 1937, Chap. IV. §6]. *F. A. Ficken* (Knoxville, Tenn.).

## *Integral Equations*

**Germay, R. H. Sur l'intégration, par approximations successives, de certains systèmes d'équations intégrales à plusieurs variables indépendantes.** Bull. Soc. Roy. Sci. Liège 21, 73–78 (1952).

The author extends some previous results [same Bull. 16, 268–275 (1947); 19, 2–8 (1950); these Rev. 11, 38; 12, 30] to systems of integral equations of the Volterra type in several independent variables. *I. A. Barnett.*

**Schmeidler, Werner. Algebraische Integralgleichungen.** Math. Nachr. 8, 31–40 (1952).

The author considers non-linear integral equations for the unknown function $y=y(s)$ $(a\leqq s\leqq b)$ of the form

$$(1)\qquad I[y]\equiv P[y]+K[y]=f(s)$$

where $P[y]=y^n(s)+a_1(s)y^{n-1}(s)+\cdots+a_{n-1}(s)y(s)$

$$(2)\quad K[y]=\sum_{\alpha+\alpha_1+\cdots+\alpha_\nu\leqq n}\int_a^b\cdots\int_a^b K_{\alpha\alpha_1\cdots\alpha_\nu}(s;t_1,\cdots,t_\nu)\times y^\alpha(s)y^{\alpha_1}(t_1)\cdots y^{\alpha_\nu}(t_\nu)dt_1\cdots dt_\nu.$$

The coefficients $a_\nu(s)$, $f(s)$, and the kernels $K_{\alpha\alpha_1\cdots\alpha_\nu}$ are real-valued continuous functions of their arguments. Moreover, it is assumed that

$$\frac{\partial P[y]}{\partial y}>0\quad\text{for}\quad y>0\quad\text{and}\quad f(s)>0\quad\text{for}\quad a\leqq s\leqq b.$$

The preceding conditions will be referred to as "assumption A". In order to formulate the additional assumptions B and C, we need the following definitions: denote by $M_{\alpha\alpha_1\cdots\alpha_\nu}$ the greater of the two numbers 0 and $\max K_{\alpha\alpha_1\cdots\alpha_\nu}(s, t_1, \cdots, t_\nu)$ for $s, t_1, \cdots, t_\nu$ ranging over the interval $[a, b]$, and set $M_{\alpha\alpha_1\cdots\alpha_\nu}-K_{\alpha\alpha_1\cdots\alpha_\nu}(s;t_1,\cdots,t_\nu)=L_{\alpha\alpha_1\cdots\alpha_\nu}(s;t_1,\cdots,t_\nu)$. Moreover, let $M[y]$ and $L[y]$ denote the expressions obtained from $K[y]$ by replacing in the integrals of (2) $K_{\alpha\alpha_1\cdots\alpha_\nu}$ by $M_{\alpha\alpha_1\cdots\alpha_\nu}$ and $L_{\alpha\alpha_1\cdots\alpha_\nu}$, respectively. Then assumption B states: There exists a continuous $y_0(s)>0$ such that $P[y_0]>f(s)+L[y_0]$.

In order to state the intermediate result which the author obtains under assumptions A and B, we make the following remarks: equation (1) can be written in the form

$$(3)\qquad P[y]+M[y]-L[y]=f(s).$$

Now $M[y]$ is a polynomial of degree $n-1$ in $y(s)$ whose coefficients $M_\alpha[y]$ $(\alpha=0, 1, \cdots, n-1)$ are constants depending on the function $y(s)$. The author considers the integral equation obtained from (3) if the $M_\alpha[y]$ are replaced by arbitrary non-negative constants $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$

subject only to the condition $\lambda_0 < f(s)$. The intermediate result referred to above consists now in the construction of a positive solution $y(s) = y(s; \lambda_0, \lambda_1, \cdots, \lambda_{n-1})$ of this new integral equation.

We are finally in a position to formulate the last assumption C: There exist non-negative constants $c_0, c_1, \cdots, c_{n-1}$ with the following property: if $y_\alpha(s) = y(s; \lambda_0, \cdots, \lambda_{n-1})$ for $\lambda_\alpha = c_\alpha$, $\lambda_\nu = 0$ for $\nu \neq \alpha$, then $M_\alpha[y_\alpha(s)] \leqq c_\alpha$. This assumption makes an application of Brouwer's fixed-point theorem possible, and thus enables the author to prove his main result: Under assumptions A, B, and C, the equation (1) has a positive continuous solution $y(s)$. The author notes that assumption B is automatically satisfied in each of the following two cases: (i) $m \geqq n$; (ii) the interval $[a, b]$ is small enough and (1) is of the Volterra type. As examples, two applied problems are treated in the second part of the paper.

*E. H. Rothe* (Ann Arbor, Mich.).

**Zenhen, O. On existence and uniqueness of solutions of integro-differential equations.** Doklady Akad. Nauk SSSR (N.S.) 86, 229–230 (1952). (Russian)

The equation

$$y^{(n)}(x) = f\left(x, y(x), \cdots, y^{(n-1)}(x), \lambda\int_a^b K(x, t, y(t), \cdots, y^{(m)}(t))dt\right)$$

$n \geqq m$, has a unique solution $y(x)$ satisfying the conditions $y^{(k)}(a) = y_k$ $(k = 0, \cdots, n-1)$ if $\lambda$ is sufficiently small and the functions $f$, $K$ satisfy certain Lipschitz conditions. The proof is immediate.

*M. Golomb* (Lafayette, Ind.).

**Colombo, Serge. Sur une équation intégrodifférentielle non linéaire.** C. R. Acad. Sci. Paris 235, 857–858 (1952).

The $L$-transform of the equation

$$af(x) - xf'(x) + \int_0^x f(\xi)f^{(n+1)}(x-\xi)d\xi = g(x)$$

with the conditions $f^{(k)}(0) = a_k$ $(k = 0, \cdots, n)$ is a Riccati equation and reduces to a linear equation if $g(x) \equiv 0$. An explicit solution is found for the case where $g(x) \equiv 0$, $a_0 = \cdots = a_n = 0$.

*M. Golomb* (Lafayette, Ind.).

**Fricke, Arnold. Eine nichtlineare Integralgleichung bei einem Problem der Zentralbewegung.** Math. Nachr. 8, 185–192 (1952).

Let $T$ be the periodic time when a particle of unit mass describes a circular orbit of radius $R$ under a central force $f(r)$; and let $T'$ be the time of fall from rest at distance $R$ to the centre under the same force. Then if the law of force is the inverse square, $T : T' = 4\sqrt{2}$ for any value of $R$. The question is to find whether $T : T'$ has a value independent of $R$ for other laws of force. This comes to solving the integral-equation

$$\int_0^R \frac{dr}{\sqrt{\left\{\int_r^R f(r')dr'\right\}}} = \lambda\sqrt{\left\{\frac{R}{f(R)}\right\}}.$$

If $f(r)$ is integrable at the origin, this can be transformed into

$$\text{(A)} \qquad \int_0^x \frac{y(u)du}{\sqrt{(x-u)}} = \lambda\sqrt{\left\{y(x)\int_0^x y(u)du\right\}},$$

whereas if it is integrable at infinity,

$$\text{(B)} \qquad \int_x^\infty \frac{y(u)du}{\sqrt{(u-x)}} = \lambda\sqrt{\left\{y(x)\int_x^\infty y(u)du\right\}}.$$

The integral equation (A) has a continuous spectrum $\lambda > \sqrt{\pi}$, with solutions $y = x^\mu$ $(\mu > -1)$, whereas (B) has spectrum $0 < \lambda < \sqrt{\pi}$ with solutions $y = x^{-\mu}$ $(\mu < -1)$. If $\lambda = \sqrt{\pi}$, (B) has the solution $y = e^{-x}$.

*E. T. Copson.*

**Lomax, Harvard, Heaslet, Max. A., and Fuller, Franklyn B. Integrals and integral equations in linearized wing theory.** Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 1054, ii+34 pp. (1951).

An extension and revision of NACA Tech. Note, no. 2252 (1950); these Rev. 12, 767.

*J. W. Miles.*

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## *Functional Analysis, Ergodic Theory*

**Berg, William D., et Nikodým, Otton Martin. Sur les ensembles convexes dans les espaces linéaires réels abstraits où aucune topologie n'est admise. I.** C. R. Acad. Sci. Paris 235, 1005–1007 (1952).

**Berg, William D, et Nikodým, Otton Martin. Sur les ensembles convexes dans l'espace linéaire où aucune topologie n'est admise. II. Corps convexes.** C. R. Acad. Sci. Paris 235, 1096–1097 (1952).

In the first note, the authors define some fundamental notions in the geometry of convex sets (in a real vector space $L$ without topology). The second note contains more definitions and proof of some "paradoxical" results. A convex set $E \subset L$ is a "convex body" provided there is a point $x \in L$ such that for every line $\lambda$ in $L$ which passes through $x$, $x$ is interior to $E \cap \lambda$ (in the natural topology of $\lambda$). $E$ is "linearly closed" ["linearly bounded"] provided $E \cap \lambda$ is closed [bounded] in $\lambda$ for each line $\lambda$ in $L$. The authors show that if $L$ is infinite-dimensional, then (a) for each convex body $E$ in $L$, $L$ contains a maximal linear proper subspace every one of whose translates intersects $E$; (b) $L$ contains linearly closed and linearly bounded convex bodies $E$ and $F$ for which the convex hull conv$(E \cup F)$ is not linearly closed, and also such $E$ and $F$ for which conv$(E \cup F)$ is not linearly bounded.

*V. L. Klee, Jr.* (Charlottesville, Va.).

**Phillips, R. S. On the generation of semigroups of linear operators.** Pacific J. Math. 2, 343–369 (1952).

Let $T(\xi)$, $\xi \geqq 0$, be a semi-group of linear bounded operators on a Banach space to itself, strongly continuous in $\xi$ and with $T(0) = I$ [Hille, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948; these Rev. 9, 594; the reviewer, J. Math. Soc. Japan 1, 15–21 (1948); these Rev. 10, 462]. The paper is devoted to the study of the generation of new semi-groups $S(t) = \int_0^\infty T(\xi)d_\xi\alpha(t, \xi)$, $t \geqq 0$, from $T(\xi)$. It is suggested by a similar but formal procedure due to Romanoff [Ann. of Math. (2) 48, 216–233 (1947); these Rev. 8, 520] and has a connection with stochastic processes due to Bochner [Proc. Nat. Acad. Sci. U. S. A. 35, 368–370 (1949); these Rev. 10, 720]. Thus P. Lévy's canonical form for the infinitely divisible law is extended in the following manner. Let $\alpha(t, \xi)$ be non-decreasing in $\xi$ such that the total variation $\|\alpha(t, \cdot)\| \leqq M$ for $0 \leqq t \leqq 1$, $\alpha(t_1 + t_2, \cdot) = \alpha(t_1, \cdot) \otimes \alpha(t_2, \cdot)$, $\alpha(0, \cdot) =$ the unit mass concentrated at the origin,

$$\int_0^\infty \omega(\xi)d_\xi\alpha(t, \xi) < \infty, \quad \text{and} \quad \lim_{t \downarrow 0}\int_0^\infty \exp(\lambda\xi)d_\xi\alpha(t, \xi) = 1$$

uniformly in $\lambda$ in any bounded subset of

$$\Re(\lambda)\leqq\omega_0=\lim_{\xi\to\infty}\omega(\xi)/\xi.$$

Here $\omega(\xi)$ is assumed to be lower semi-continuous, subadditive and bounded near the origin such that $\omega(0)=0$. Then we have the representation

$$t^{-1}\int_0^\infty \exp(\lambda\xi)d_\xi\alpha(t,\xi)$$
$$=m\lambda+\int_0^\infty(\exp[(\lambda-\omega_0)\xi]-1)d\psi(\xi)\quad\text{for}\quad\Re(\lambda)\leqq\omega_0,$$

where $m\geqq0$, $a$ is real, and $\psi(\xi)$ is non-decreasing such that $\int_0^1\xi d\psi(\xi)<\infty$ and $\int_1^\infty\exp[\omega(\xi)-\omega_0\xi]d\psi(\xi)<\infty$. Also, for general $S(t)$ with $\omega(\xi)=\log\|T(\xi)\|$, the infinitesimal generator $B$ of $S(t)$ is connected with the infinitesimal generator $A$ of $T(\xi)$ by the relation

$$B=mA+\int_0^\infty[\exp(-\omega_0\xi)T(\xi)-I]d\psi+aI.$$

*K. Yosida* (Osaka).

**Ramaswami, V. On a theorem of Gelfand and Hille.** J. Indian Math. Soc. (N.S.) **14**, 129–138 (1950).

The author generalizes a result of Gelfand [Mat. Sbornik N.S. **9**(51), 49–50 (1941); these Rev. **3**, 36] and Hille [Proc. Nat. Acad. Sci. U. S. A. **30**, 58–60 (1944); these Rev. **5**, 189]. See also Stone [J. Indian Math. Soc. (N.S.) **12**, 1–7 (1948); these Rev. **10**, 308] for a simplified proof. For a linear normed space $X$, the author establishes the following two theorems. (1) Let $x_\nu$, $\nu\geqq0$, be a sequence in $X$ such that $\lim_{\nu\to\infty}\|x_\nu\|^{1/\nu}=0$. Then a necessary and sufficient condition that $x_{n+1}=\theta$ for $n\geqq k\geqq0$ is that

$$\limsup_{p\to\infty}\|\sum_{\nu=0}^{p}(-1)^\nu(n,\nu)x_\nu\|=O(n^k)$$

as $|n|\to\infty$ over any assigned endless arithmetic progression of integers. (2) Set $x(n,p)=\sum_{\nu=0}^p(-1)^\nu(n,\nu)x_\nu$ and suppose that $\limsup_{\nu\to\infty}\|x_\nu\|^{1/\nu}<\infty$. Then (i) a necessary and sufficient condition for $\lim_{\nu\to\infty}\|x_\nu\|^{1/\nu}=0$ without every $x_\nu=\theta$ is that $\limsup_{\nu\to\infty}\|x(\nu,\nu)\|^{1/\nu}=1=\limsup_{\nu\to\infty}\|x(\nu)\|^{1/\nu}$ where $x(\nu)=\sum_{p=0}^\nu(\nu,p)x_p$; (ii) a necessary and sufficient condition that the conclusion of (1) hold is that $\|x(\nu,\nu)\|+\|x(\nu)\|=O(\nu^k)$ as $\nu\to\infty$, or $\|\sum_{p=0}^\nu(\nu,2p+1)x_p\|=O(\nu^{2k+1})$ as $\nu\to\infty$, or $\|\sum_{p=0}^\nu(\nu,2p)x_p\|=O(\nu^{2k})$ as $\nu\to\infty$. *R. S. Phillips.*

**Wermer, John. The existence of invariant subspaces.** Duke Math. J. **19**, 615–622 (1952).

Let $B$ be a complex Banach space and $T$ a bounded linear operator on $B$. By a "non-trivial invariant subspace" of $B$ with respect to $T$ the author means a proper non-trivial closed subspace which is mapped into (not necessarily onto) itself by $T$. The author's main result is that, if $T$ has a bounded inverse, and $\|T^n\|\leqq d_n$ for all $n$, positive, negative, or zero, where $d_n=d_{-n}\geqq1$, $\sum_0^\infty(\log d_n)/(1+n^2)$ is convergent, and $d_n$ is non-decreasing and $(\log d_n)/n$ decreasing for positive $n$, then $T$ has a non-trivial invariant subspace in $B$ provided its spectrum does not reduce to a single point. He further shows that, if $T$ has a bounded inverse and $\|T^n\|=O(|n|^k)$, then $T$ has a non-trivial invariant subspace even if its spectrum does reduce to a single point (provided, of course, $B$ is at least two-dimensional). *A. F. Ruston.*

**Vainberg, M. M. Some questions of the differential calculus in linear spaces.** Uspehi Matem. Nauk (N.S.) **7**, no. 4(50), 55–102 (1952). (Russian)

In the first section the function theory of operators and, especially, functionals in a Banach space $E$ with dual $E'$ (the duality being expressed by $(x', x)$) is developed. Continuity, compactness (=mapping of bounded onto relatively compact sets), complete continuity (=strong continuity+compactness), complete compactness (=compactness+uniform continuity) are defined and related to one another and to other functional properties.

The Gateaux differential $Df(x, h)$ of a functional $f(x)$ is defined as the derivative of $f(x+th)$ at $t=0$ if this exists. If it is a linear functional of $h$, i.e., if $Df(x, h)=(F(x), h)$, write $F(x)=\operatorname{grad} f(x)$. If (A) $f(x+h)-f(x)=V(x, h)+\omega(x, h)$, then, if $V$ is linear in $h$ and $\|\omega(x, h)\|$ is $o(h)$, $V(x, h)=df(x, h)$ is the Fréchet differential of $f$. If, in (A), $V$ is bounded in some neighbourhood of $h=0$, it is called a bounded differential. If, in addition $\omega(x, h)$ is locally uniform, i.e., if for each $\epsilon>0$ there are positive $\delta$, $\eta$ such that $\|\omega(x, h)\|<\epsilon\|h\|$ if $\|h\|<\delta$ and $\|x-x_0\|<\eta$, then $V$ is called a uniform differential. Relationships between these differentials are discussed. For example, if $F(x)$ exists in a neighbourhood of $x$ and is continuous at $x$, the Fréchet differential exists at $x$. A uniform differential is a Fréchet differential.

An operator in $E$ to $E'$ is called potential if it is the gradient of some functional. A sufficient condition for $F$ to be potential, due to Kerner [Ann. Mat. Pura Appl. (4) **10**, 145–164 (1932)], is given.

If $S(r)$ denotes the sphere $\|x\|\leqq r$, then for $F$ to be continuous in $S(r)$ it is necessary and sufficient that $df(x, h)$ have a locally uniform remainder and that $F$ be locally bounded in $S(r)$. If $F$ is bounded in $S(r+\alpha)$ for some $\alpha>0$, then for it to be uniformly continuous in $S(r)$ it is necessary and sufficient that $df(x, h)$ have a locally uniform remainder in $S(r)$. Under the further assumption that the unit sphere in $E$ is weakly compact, it is shown that for $F$ to be weakly continuous in $S(r)$ it is necessary that it be completely compact and sufficient that it be uniformly continuous in $S(r)$ and compact in $S(r+\alpha)$ for some $\alpha>0$. If $F$ is compact in $S(r+\alpha)$, a necessary and sufficient condition that it be weakly continuous in $S(r)$ is that $df(x, h)$ have a locally uniform remainder.

Conditions for compactness of the gradient in spaces with a countable biorthogonal basis are given. Finally a number of interesting examples applying the theory to nonlinear functionals of standard forms in the main function spaces are given. *J. L. B. Cooper* (Cardiff).

**Pol'skiĭ, N. I. Some generalizations of B. G. Galerkin's method.** Doklady Akad. Nauk SSSR (N.S.) **86**, 469–472 (1952). (Russian)

Let $A(\lambda)$ be completely continuous in the Banach space $E$ for each complex $\lambda$ of a domain $D$, and analytic in the uniform topology of operators. For $f$ in $E$, consider the equation $x=A(\lambda)x+f$. Solutions of this equation were approximated by Galerkin as follows: Let $\{\varphi_i; \Phi_i\}$ be a basis for $E$, $\Phi_i(\varphi_j)=\delta_{ij}$ and let $C_i^{(n)}$ be determined by the equations $C_i^{(n)}-A(\lambda)C_i^{(n)}-\Phi_i(f)=0$, $i=1, 2, \cdots, n$. Then the elements $x_n=\sum_{i=1}^nC_i^{(n)}\varphi_i$ are the approximations to the solution of the given equation. The author's generalization consists in choosing two bases for $E$, $\{\varphi_i; \Phi_i\}$ and $\{\psi_i; \Psi_i\}$ and using the approximants $x_n=\sum_{i=1}^nC_i^{(n)}\psi_i$ where $C_i^{(n)}$ are solutions of $\Phi_i(x_n-A(\lambda)x_n-f)=0$. The underlying assumption is that there is a constant $C$ for which $\|x\|\leqq C\|P_nx\|$

where $P_n x = \sum_{i=1}^n \Phi_i(x)\varphi_i$. Theorems: 1. If the assumption is valid for both bases and if $\lambda$ is not an eigen-value of $A(\lambda)$ (i.e., $x = A(\lambda)x$ implies $x=0$), then after a certain $n$, the $x_n$ are uniquely defined and they converge in $E$ to a solution of the equation. 2. The eigen-values of $A(\lambda)$ are precisely the set of limit points of the eigen-values $\lambda_n$ (i.e., $(P_n - P_n A(\lambda_n))x_n = 0$ has a nontrivial solution). 3. Let $\lambda_n \to \lambda_0$ and $x_n$ be the corresponding eigen-vectors. Then one can choose a subsequence of the $x_n$ converging to some $x$ in $E$, and this $x$ will be an eigen-vector of $A(\lambda)$ for $\lambda = \lambda_0$. 4. If $x_0$ is in an invariant subspace of $A(\lambda)$, for $\lambda = \lambda_0$, one can find a sequence of linear combinations of elements of the invariant subspaces $\{x_n \mid (P_n - P_n A(\lambda_n))^k x_n = 0 \text{ for some } k\}$ which converges to $x_0$ and for which the corresponding $\lambda_n \to \lambda_0$.

If the basic assumption is dropped, counterexamples to the above can be produced. The ideas are applied to Hilbert space problems in which the method of least squares is used to obtain approximants, i.e., the expression $\|x_n - A(\lambda)x_n - f\|^2$ is minimized on a finite-dimensional subspace.
*B. R. Gelbaum* (Minneapolis, Minn.).

**Mimura, Yukio. On a theorem of O. Toeplitz.** Sci. Papers Coll. Gen. Ed. Univ. Tokyo 2, 9–11 (1952).

If $A$ is a closed linear transformation whose domain $\mathfrak{D}$ is dense in a Hilbert space, then a necessary and sufficient condition that there exist a bounded linear operator $B$ such that $BAf = f$ for all $f$ in $\mathfrak{D}$ is that there exist a positive constant $m$ such that $\|Af\| \geqq m\|f\|$ for all $f$ in $\mathfrak{D}$. The proof is the same as for bounded operators. Familiar considerations involving adjoints convert this left theorem into a right theorem. (In relations (1) and (2) of the paper, the sign $>$ should be $\geqq$.) *P. R. Halmos* (Chicago, Ill.).

**Sobolev, V. I. On a property of self-adjoint operators in Hilbert space.** Uspehi Matem. Nauk (N.S.) 7, no. 4(50), 169–172 (1952). (Russian)

Let $H$ be a Hilbert space and let $\mathfrak{R}_A$ be the ring of all bounded self-adjoint operators which commute with a bounded self-adjoint operator $A$. Then, under the natural ordering, $\mathfrak{R}_A$ is a lattice-ordered (real) linear space in which every bounded set admits least upper and greatest lower bounds. This extends a result announced by M. H. Stone [Proc. Nat. Acad. Sci. U. S. A. 26, 280–283 (1940); these Rev. 1, 338]. *E. Hewitt* (Seattle, Wash.).

**Povzner, A. Ya. On M. G. Kreĭn's method of directing functionals.** Učenye Zapiski Har'kov. Gos. Univ. 28, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) 20, 43–52 (1950). (Russian)

If $A$ is a linear operator with dense domain in a Hilbert space $H$, a system of additive functionals $\Phi_j(f, \lambda)$ $(j = 1, 2, \cdots, p)$ is called a directing system of functionals for $A$ if they are defined for all real $\lambda$ and $f$ in a dense linear manifold $L$ of $H$, and if, for each $f \in L$ and $\lambda$, $|\Phi_j(f, \lambda) - \Phi_j(f, \mu)| \leqq M_\lambda(f)|\lambda - \mu|$, if $Af - \lambda_0 f = f_0$ has a solution for $f_0 \in L$ if and only if $\Phi_j(f_0, \lambda_0) = 0$ $(j = 1, 2, \cdots, p)$; and if for any interval $(-a, a)$ there are $p$ elements $l_1, l_2, \cdots, l_p$ in $L$ such that $\det \Phi_j(l_k, \lambda) = 0$ for $-a \leqq \lambda \leqq a$. Let $A$ be assumed self-adjoint. $H$ is then isomorphic to a space $\tilde{H}$ of functions on a space $\mathfrak{M}$ of points $M = (k, \lambda)$ where $\lambda$ is in the spectrum of $H$ with multiplicity $m$, $k = 1, 2, \cdots, m$; each $f \in H$ corresponds to $f(M)$ in $\tilde{H}$ with

$$\|f\|^2 = \|f(M)\|^2 = \int_{\mathfrak{M}} f(M)d\omega(M)$$

where $\omega$ is a Borel measure, and the subspaces of $H$ corresponding to constant $k$ reduce $A$ so that in them

$$Af(M) = \varphi(M)f(M) = \lambda f(M).$$

It is shown, generalising a formula of M. G. Kreĭn [C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 3–6 (1946); these Rev. 8, 277], that there exist functions $\tau_1(M), \cdots, \tau_p(M)$ such that for each $g \in L$, $g(M) = \sum_j \tau_j(M)\Phi_j(g, \varphi(M))$.

For $H$ taken as $L^2(-\infty, \infty)$ and

$$\Phi_j(f, \lambda) = \int_{-\infty}^{\infty} f(x)\varphi_j(x, \lambda)dx$$

the inversion formulae

$$f(M) = \lim_{n\to\infty} \sum \tau_j(M) \int_{-n}^{n} \varphi_j(x, \varphi(M))f(x)dx,$$

$$f(x) = \int_{\mathfrak{M}} g(M) \sum_{j=1}^{p} \bar{\tau}_j(M)\bar{\varphi}_j(x, \varphi(M))d\omega(M)$$

are proved. If $A$ has a system of $p$ directing functionals, its spectral multiplicity does not exceed $p$. *J. L. B. Cooper.*

**Sunouchi, Haruo. On rings of operators of infinite classes. I.** Proc. Japan Acad. 28, 9–13 (1952).

**Sunouchi, Haruo. On rings of operators of infinite classes. II.** Proc. Japan Acad. 28, 330–335 (1952).

Extensions of work of Murray and von Neumann [Ann. of Math. (2) 37, 116–229 (1936)] and Dixmier [Ann. Sci. Ecole Norm. Sup. (3) 66, 209–261 (1949); these Rev. 11, 370] which, however, are contained in later work of Dixmier [C. R. Acad. Sci. Paris 230, 607–608 (1950); these Rev. 11, 524; and Compositio Math. 10, 1–55 (1952)]. In Part I the existence of a center-valued trace on (suitable subsets of the) finite projections in a ring is obtained; in Part II further properties of the trace are developed. *I. E. Segal.*

irreducible

**Rickart, C. E. Representation of certain Banach algebras on Hilbert space.** Duke Math. J. 18, 27–39 (1951).

Let $\mathfrak{X}$ be an infinite-dimensional real (resp. complex) Banach space, and $\mathfrak{A}$ an ~~indecomposable~~ algebra of bounded operators on $\mathfrak{X}$. Suppose that $\mathfrak{A}$ is a Banach algebra under some norm, that $\mathfrak{A}$ contains finite-dimensional operators, and that there exists a mapping $T \to T^*$ from $\mathfrak{A}$ into $\mathfrak{A}$ such that $(T^*)^* = T$, $(ST)^* = T^*S^*$, and $TT^* = 0$ implies $T = 0$. The author now proves that there exists a real (resp. complex) inner product $(x, y)$ on $\mathfrak{X}$ such that $(Tx, y) = (x, T^*y)$ for all $T$ in $\mathfrak{A}$, every $T$ in $\mathfrak{A}$ is bounded relative to the norm $|x| = (x, x)^{1/2}$ on $\mathfrak{X}$, and the given norm on $\mathfrak{A}$ is greater than a constant times the operator norm relative to $|x|$. If, moreover, $\|TT^*\| \geqq \alpha\|T\|^2$ $(\alpha > 0)$ for all $T$ in $\mathfrak{A}$, or if $\mathfrak{A}$ is dense in the sense of Jacobson and strongly irreducible (i.e., there exists a $\beta > 0$ such that, for $x, y$ in $\mathfrak{X}$, $x \neq 0$, there exists a $T$ in $\mathfrak{A}$ such that $Tx = y$ and $\|T\| \cdot \|x\| \leqq \beta\|y\|$), then the given norm on $\mathfrak{X}$ is equivalent to the norm $|x|$. These results generalize a theorem of Kakutani and Mackey [Ann. of Math. (2) 45, 50–58 (1944); these Rev. 5, 146] (resp., [Bull. Amer. Math. Soc. 52, 727–733 (1946); these Rev. 8, 31]), who obtained these conclusions under the assumption that $\mathfrak{A}$ is the algebra of all bounded operators on $\mathfrak{X}$, and that also $(S+T)^* = S^* + T^*$. *E. Michael* (Chicago, Ill.).

**Šilov, G. E. On homogeneous rings of functions on the torus.** Doklady Akad. Nauk SSSR (N.S.) 82, 681–684 (1952). (Russian)

In this note, special cases of a theory presented earlier are worked out [see Uspehi Matem. Nauk (N.S.) 6, no.

1(41), 91–137 (1951); these Rev. **13**, 139]. A complete classification is given for all homogeneous algebras $R$ of type $C$ of complex functions on the torus $T$ such that $D_{11} \subset R \subset C$, where $D_{11}$ is the algebra of all functions on $T$ with continuous first partial derivatives and $C$ is the algebra of all continuous functions. The torus is parametrized by variables $(s, t)$, $-\pi \leqq s, t \leqq \pi$. Apart from $D_{11}$ and $C$, essentially the only possibilities are the following: (A) The algebra of all continuous complex functions $f$ on $T$ having a continuous partial derivative $\partial f/\partial s$, and with the norm $\|f\| = \sup \{|f| + |\partial f/\partial s|\}$; (B) the completion of the ring of all polynomials $P(s, t) = \sum \alpha_{mn} e^{ims+int} = u(s, t) + iv(s, t)$ under the norm

$$\|P\| = \sup_{s,t} \left\{ |P(s, t)| + \left| \left( \frac{\partial u}{\partial s} + \frac{\partial v}{\partial t} \right) + i \left( \frac{\partial v}{\partial s} - \frac{\partial u}{\partial t} \right) \right| \right\}.$$

(The algebra (B) is identified as the algebra of all vectorially smooth functions, which the author has described elsewhere [Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 176–184 (1951); these Rev. **14**, 374]. All other function rings $R$ as described are obtained from (A) by replacing $\partial f/\partial s$ by $\partial f/\partial \alpha$, an arbitrary directional derivative, or from (B) by locally affine changes of co-ordinates. *E. Hewitt.*

**Leibenzon, Z. L. On the ring of continuous functions on a circle.** Uspehi Matem. Nauk (N.S.) 7, no. 4(50), 163–164 (1952). (Russian)

Denote by $\Gamma$ the boundary of the unit circle in the complex plane and let $C$ denote the Banach algebra of all complex-valued continuous functions $f$ defined on $\Gamma$ with $\|f\| = \max_{|t|=1} |f(t)|$ as norm. Let $A$ be the smallest closed subalgebra of $C$ which contains the function $f(t) = t$. Then $A$ consists of boundary values of functions $f(z)$ analytic for $|z| < 1$ and continuous for $|z| \leqq 1$. It is obvious that $A$ is a proper subalgebra of $C$. For $\varphi \in C - A$, denote by $(A, \varphi)$ the smallest closed subalgebra of $C$ which contains $A$ and $\varphi$. Whether or not $(A, \varphi) = C$ for all $\varphi \in C - A$ is not known. The author considers conditions on $\varphi$ which give $(A, \varphi) = C$. The first condition is that $\varphi$ be a non-constant real function. This is proved using (unavailable) results of Šilov [On rings of functions with uniform convergence, Ukrain. Mat. Žurnal 3 (1951)]. Next let $\varphi \in C - A$ and define

$$\frac{1}{2\pi i} \int_\Gamma \frac{\varphi(t)dt}{t - z} = \begin{cases} F_i(z), & |z| < 1 \\ F_e(z), & |z| > 1. \end{cases}$$

Assume that for $t \in \Gamma$, $F_i(z) \to \varphi_i(t)$ as $z \to t$ with $|z| < 1$ and $F_e(z) \to \varphi_e(t)$ as $z \to t$ with $|z| > 1$, where $\varphi_i$ and $\varphi_e$ belong to $C$. If $\varphi = \varphi_i - \varphi_e$, then $\varphi$ is called a "jump function". The second result is that $(A, \varphi) = C$ if $\varphi$ is a jump function in $C - A$. It has been shown by Privalov [Boundary properties of analytic functions, Moscow-Leningrad, 1950, p. 197; these Rev. **13**, 926] that a function $\varphi$ which satisfies a Lipschitz condition of order $\alpha$ $(0 < \alpha \leqq 1)$ is a jump function.

*C. E. Rickart* (New Haven, Conn.).

**Heider, L. J. Lattice ordering on Banach spaces.** Proc. Amer. Math. Soc. 3, 833–838 (1952).

The author translates properties of a linear partial ordering of a real Banach space $B$ into properties of the subset $B^+$ of $B$ consisting of all $b \in B$ with $b \geqq 0$ (so that $b \geqq c$ is equivalent to $b - c \in B^+$, and $(B^+ + c)$ is the set of all $b \in B$ with $b \geqq c$). Thus, it is known that $B$ is a linear lattice if and only if $b^+ = b \vee 0$ exists in the usual sense for every $b \in B$, and this can be expressed by saying that for every $b \in B$ there exists $b^+ \in B$ such that $(B^+ + b) \cap B^+ = (B^+ + b^+)$. A set $B^+$ with this property he calls $B^{+\vee}$. The existence of an element $e \in B^+$ with $\|e\| = 1$, and such that $\|b\| \leqq 1$ if and only if $-e \leqq b \leqq e$ can be expressed by saying that there exists $e \in B^{+\vee}$ with $\|e\| = 1$, and such that $\|b\| \leqq 1$ if and only if $e + b$ and $e - b$ are both in $B^{+\vee}$. A subset $B^{+\vee}$ for which such an $e$ exists is called $B^{+\vee e}$. It follows then from a result of M. and S. Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 427–430 (1940); these Rev. **2**, 222] that a real Banach space $B$ is equivalent to the space of all real continuous functions on some bicompact Hausdorff space if and only if it contains a subset $B^{+\vee e}$.

The author applies a similar treatment to the abstract $(L)$-spaces of Kakutani [Ann. of Math. (2) **42**, 523–537 (1941); these Rev. **2**, 318]. A $T$-set $T$ is a subset of $B$ maximal with respect to the property that $\|\sum_{i=1}^n b_i\| = \sum_{i=1}^n \|b_i\|$ for any finite set of elements $b_i \in T$. A Banach space can be given a linear partial ordering under which it is an abstract $(L)$-space if and only if it contains a closed $T$-set $T$ with the properties that for every $b \in B$ there is a $b^+$ such that $(T + b) \cap T = (T + b^+)$, and that $\|b - c\| = \|b + c\|$ whenever $(T - b) \cap (T - c) = T$.

Errata: p. 835, l. 12: a symbol has been inverted at the end of the line; p. 836, l. 4: the last expression should read "$\|b - c\|$"; p. 836, l. 18: "$\|b\|^+$" should read "$\|b^+\|$".

*A. F. Ruston* (London).

**Haefeli, Hans Georg. I funzionali lineari delle funzioni analitiche di una variabile quaternionale.** Rend. Accad. Naz. dei XL (4) 2, 65–110 (1951).

Soit $E^4$ l'espace des vecteurs $x = \sum_0^3 x_k i_k$, à composantes $x_k$ réelles, avec la règle de multiplication qui le fait devenir l'algèbre des quaternions. Considérons une fonction $w(x) = \sum_0^3 u_j(x) i_j$, définie dans un ouvert $H$ di $E^4$ et dont les composantes $u_j(x)$ soient des fonctions réelles admettant des dérivées partielles continues par rapport aux $x_k$; en posant

$$D = \sum_{k=0}^{3} \frac{\partial}{\partial x_k} i_k \quad \text{et} \quad u_k \frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k} u_k,$$

on dit que $w(x)$ est analytique droite [analytique gauche], si l'on a $wD = 0$ [$Dw = 0$]. À partir de cette définition, R. Fueter et ses élèves ont développé une théorie des fonctions $w(x)$ analytiques, semblable à la théorie classique des fonctions analytiques d'une variable complexe.

L'auteur construit, pour les fonctions analytiques en question, l'analogue de la théorie des fonctionnelles analytiques de Fantappiè. Il considère l'espace compact $S^4$ qui résulte de $E^4$ par adjonction d'un point à l'infini. La fonction $w(x)$ est dite régulière au point $\infty$, s'il existe un voisinage $V$ de ce point tel que $w(x)$ soit bornée dans $V$ et analytique dans $V \cap E^4$. Cela étant, l'auteur définit l'espace fonctionnel $S^Q$ comme l'ensemble des fonctions $w(x)$ définies dans des ouverts $R$ de $S^4$ $(R \neq 0, R \neq S^4)$, régulières et analytiques droites dans ces ensembles, et s'annulant pour $x = \infty$, si $\infty \in R$. (Pour les fonctions analytiques gauches on a une théorie isomorphe.) Ensuite, il introduit les concepts de "voisinage", "région linéaire", "ligne analytique", et "fonctionnelle analytique" à peu près comme dans la théorie classique; on doit signaler ces changements: la représentation $w(\alpha, x)$ d'une ligne analytique de $S^Q$ est une fonction analytique gauche du paramètre $\alpha$; une fonctionnelle analytique $F[w]$ devient une fonction analytique gauche de $\alpha$ sur chaque ligne analytique $w(\alpha, x)$. D'ailleurs, les résultats et les techniques de démonstration sont analogues à ceux de la théorie classique. Une région linéaire $(A)$ est constituée par les éléments de $S^Q$ dont les domaines $R$ contiennent

l'ensemble fermé $A$. L'auteur obtient pour les fonctionnelles analytiques définies dans $(A)$ l'expression générale suivante:

$$(1)\qquad F[w]=\frac{1}{2\pi^2}\int_C w(\xi)(\mathfrak{n}dr)v(\xi),$$

où $C$ est une hypersurface de $E^4$ contenue dans le domaine de $w(x)$, orientable au moyen d'une normale $\mathfrak{n}$ (unitaire) dirigée vers $A$; $dr$ est l'élément différentiel de $C$ et $v(\xi)$ une fonction analytique gauche de domaine $S^4-A$ donnée par

$$v(\xi)=F_x\left[\frac{\xi-\bar{x}}{n^2(\xi-x)}\right],$$

en posant $\bar{x}=x_0i_0-\sum_1^3x_ki_k$, $n(x)=x\bar{x}$.

Cette théorie trouve application dans l'étude des fonctionnelles linéaires de fonctions analytiques ordinaires de deux variables complexes. Elle permet de représenter au moyen d'une formule du type (1) une vaste classe de ces dernières fonctionnelles, même dans le cas où la formule de Fantappiè n'est plus applicable, l'ensemble caractéristique de la région linéaire n'étant pas le produit cartésian de deux sous-ensembles de la droite complexe.

*J. Sebastião e Silva* (Lisbonne).

## *Theory of Probability*

**Karhunen, Kari. Linear methods in the calculus of probabilities.** Trabajos Estadística **3**, 59–137 (1952). (Spanish. English summary)

Translated from Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **37** (1947); these Rev. **9**, 292.

**Hammersley, J. M. Tauberian theory for the asymptotic forms of statistical frequency functions.** Proc. Cambridge Philos. Soc. **48**, 592–599 (1952).

If $\varphi_n(x)$ is real (measurable) convex on the reals for $a_n<x<b_n$ $(n=1, 2, \cdots)$ and $a'=\limsup a_n<\liminf b_n=b$ and $\varphi(x)=\lim_{n\to\infty}\varphi_n(x)$ exists for

$$\liminf a_n=a<x<b'=\limsup b_n,$$

then $\varphi(x)$ is convex for $a<x<b'$ and

$$\lim \varphi_n'(x-0)=\lim \varphi_n'(x+0)=\varphi'(x)$$

whenever the latter exists $(a'<x<b)$. This may be applied to sequences of unimodal probability density functions or discrete probability functions on the integers: if the corresponding suitably normalized distribution functions approach a limit $F$, then (under weak restrictions) the normalized density or probability functions will approach that corresponding to $F$, except perhaps at the mode of the latter, where the author shows by example the various possible behaviors. A corresponding result holds for multimodal functions, the modes and nadirs now becoming the exceptional points. Applications are given, several being to cases where previous proofs of the final result were quite tedious.

*J. Kiefer* (Ithaca, N. Y.).

**Fortet, Robert, et Mourier, Edith. Loi des grands nombres et théorie ergodique.** C. R. Acad. Sci. Paris **234**, 699–700 (1952).

Let $\mathfrak{X}$ be a separable Banach space and $L^\alpha(\mathfrak{X})$ $(1\leqq\alpha<\infty)$ be the $L^\alpha$-space of all weakly measurable $\mathfrak{X}$-valued random variables $X(\omega)$ (defined on a probability space $\Omega=\{\omega\}$) such that $E(\|X(\omega)\|^\alpha)=\int_\Omega\|X(\omega)\|^\alpha d\omega<\infty$, where $E$ denotes the expectation. Let further $\{X_n(\omega)\,|\,n=1, 2, \cdots\}$ be an independent sequence of random variables from $L^\alpha(\mathfrak{X})$. The following results are announced: (i) if $X_n(\omega)$ has the same distribution for $n=1, 2, \cdots$, then there exists a random variable $Y(\omega)\in L^\alpha(\mathfrak{X})$ (same $\alpha$) such that

$$\lim_{n\to\infty} E(\|n^{-1}\sum_{k=1}^n X_k-Y\|^\alpha)=0$$

(the law of large numbers in the mean of power $\alpha$); (ii) if $E(\|X_n\|^\alpha)$ is bounded in $n$ and if $E(X_n)=\int_\Omega X_n(\omega)d\omega=0$ for $n=1, 2, \cdots$, then $\lim_{n\to\infty}E(\|n^{-1}\sum_{k=1}^nX_k\|^\beta)=0$ for any $\beta$ with $0<\beta<\alpha$. (i) was previously announced by Mourier [C. R. Acad. Sci. Paris **232**, 923–925 (1951); these Rev. **12**, 616] under the additional assumption that $\mathfrak{X}$ is reflexive and $\alpha>1$.

*S. Kakutani* (New Haven, Conn.).

**Freudenthal, Hans. Gambling with a poor chance of gain.** Nederl. Akad. Wetensch. Proc. Ser. A. **55**=Indagationes Math. **14**, 433–438 (1952).

Let $x_n$ be mutually independent variables with

$$\Pr\{x_n=0\}=1-a_n^{-1},\quad \Pr\{x_n=a_n\}=a_n^{-1}.$$

In order that $\Pr\{|n^{-1}\sum_1^nx_k-1|<\epsilon\}\to1$, it is necessary and sufficient that $a_n=O(a_n)$. For the case $a_n=n$ the author determines the limiting distribution of $n^{-1}\sum_1^nx_k$.

*W. Feller* (Stockholm).

**Freudenthal, Hans. A limit free formulation of the weak law of large numbers.** Nederl. Akad. Wetensch. Proc. Ser. A. **55**=Indagationes Math. **14**, 427–432 (1952).

The usual formulation of the weak law of large numbers for independent random variables states that

$$\Pr\{|a_n^{-1}S_n|>\epsilon\}\to0$$

if and only if certain other expressions tend to zero. The author replaces this formulation by a statement that two sets of inequalities are equivalent. [He informs the reviewer that the formula $x^k=0$ on p. 427 should read $x^K=$some point in $K$, and that his account of preceding work is in error and should be deleted.]

*W. Feller.*

**Yosida, Kôsaku. On Brownian motion in a homogeneous Riemannian space.** Pacific J. Math. **2**, 263–270 (1952).

Let $R$ be an $n$-dimensional, orientable, infinitely differentiable Riemannian space such that the group $G$ of isometric transformations $S^*$ of $R$ onto $R$ constitutes a Lie group transitive on $R$. Suppose furthermore that the subgroup of $G$ leaving a point $x$ of $R$ invariant is compact. Consider a Markov process on $R$ which is temporally and spatially homogeneous, that is, it has transition probabilities with the property $P(t, x, E)=P(t, S^*x, S^*E)$ for each $S^*\in G$. The author shows that the familiar continuity condition

$$t^{-1}\int_{d(x,y)>\epsilon} P(t, x, dy)\to0$$

implies

$$\limsup t^{-1}\int_R\frac{d^2(x, y)}{1+d^2(x, y)}P(t, x, dy)<\infty$$

as $t\to0$ with $d(x, y)$ denoting the distance between $x$ and $y$. From this the author concludes that for every $f(x)$ satisfying certain regularity conditions one has

$$t^{-1}\left\{\int_Rf(y)P(t, x, dy)-f(x)\right\}\to a^i(x)f_i+b^{ij}(x)f_{ij},$$

subscripts denoting partial derivatives. This is, essentially, the backward diffusion equation. The point is that no artificial differentiability conditions on $P$ are imposed.
*W. Feller* (Stockholm).

**Ramakrishnan, Alladi. A note on Janossy's mathematical model of a nucleon cascade.** Proc. Cambridge Philos. Soc. **48**, 451–456 (1952).

The author discusses in detail the correspondence between the notations and methods used by himself [same Proc. **46**, 595–602 (1950); these Rev. **14**, 296] and Jánossy [Proc. Roy. Irish Acad. Sect. A. **53**, 181–188 (1950); these Rev. **13**, 569]. He then treats a nucleon cascade which differs from the electron-photon cascade treated by him [loc. cit.] in that only one type of particle is present.
*W. Feller* (Stockholm).

**Jánossy, L. On the generalization of Laplace transform in probability theory.** Acta Math. Acad. Sci. Hungar. **2**, 177–184 (1951). (Russian summary)

This is the English version of a paper by the same author [Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 343–350 (1951); these Rev. **13**, 957]. *E. Lukacs.*

**Jánossy, L. Study of a stochastic process arising in the theory of the electron multiplier.** Acta Math. Acad. Sci. Hungar. **2**, 165–176 (1951). (Russian summary)

This is the English version of a paper by the same author [Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 357–367 (1951); these Rev. **13**, 957]. *E. Lukacs.*

**Takács, Lajos. Occurrence and coincidence phenomena in case of happenings with arbitrary distribution law of duration.** Acta Math. Acad. Sci. Hungar. **2**, 275–298 (1951). (Russian summary)

This is a slightly modified English version of an earlier paper by the same author [Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 371–386 (1951); these Rev. **13**, 956].
*E. Lukacs* (Washington, D. C.).

**Linnik, Yu. V. Remarks concerning the classical derivation of Maxwell's law.** Doklady Akad. Nauk SSSR (N.S.) **85**, 1251–1254 (1952). (Russian)

The derivation of Maxwell's law of distribution is usually based on redundant physical assumptions. The author offers three sets of mathematical assumptions. The first two are based on known characterizations of the normal distribution; the third one on a recent result by the author [same Doklady (N.S.) **83**, 353–355 (1952); these Rev. **14**, 60]. The author seems unaware of the work of Kac [Amer. J. Math. **61**, 726–728 (1939); these Rev. **1**, 62].
*K. L. Chung* (Ithaca, N. Y.).

**✶Solodovnikov, V. V. Vvedenie v statističeskuyu dinamiku sistem avtomatičeskogo upravleniya. [Introduction to the statistical dynamics of systems of automatic regulation.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, Leningrad, 1952. i+367 pp. 11.45 rubles.

Topics discussed include: frequency characteristics, transfer and transient functions of dynamical systems; a résumé of the theory of probability; stationary random processes; passage of random signals through dynamical systems and calculation of the mean square value of random variables; on the method of approximating spectral density functions by rational fractional functions; condition determining minimum mean square error and formula for optimal transfer function; optimal statistical prediction, smoothing, and differencing; calculation of optimal transfer in the case of finite time of observation. Recent U. S. literature is cited. References are made among others to work of Wiener, Bode, and especially to a Russian translation of "Theory of servomechanisms" by James, Nichols, and Phillips [McGraw-Hill, New York, 1947; these Rev. **11**, 517].
*N. Levinson* (Cambridge, Mass.).

## Mathematical Statistics

**✶Rao, C. Radhakrishna. Advanced statistical methods in biometric research.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1952. xvii+390 pp. $7.50.

The author's preface states that his object "is to present a number of statistical techniques, keeping in view the requirements of both the student who questions the basis of a particular method employed and the practical worker who seeks a recipe for the reduction of his data". Chapter titles: I. Algebra of vectors and matrices. II. Theory of distributions. III. The theory of linear estimation and tests of hypotheses. IV. The general theory of estimation and the method of maximum likelihood. V. Large sample tests of hypotheses with applications to problems of linear estimation. VI. Tests of homogeneity of variances and correlations. VII. Tests of significance in multivariate analysis. VIII. Statistical inference applied to classificatory problems. IX. The concept of distance and the problem of group constellations.

It is remarkable that so much ground has been covered in 390 pages, especially since the author provides both theory and a number of problems (mostly biometric) worked out in full to illustrate computational procedures. He has achieved this by three methods. Firstly, he takes for granted a considerable knowledge of the theory of probability (so that Chapter II deals only with the special distributions involved in the statistical procedures of later chapters). Secondly, he has endeavoured "to integrate a large collection of computational schemes into consistent patterns". Thirdly, he writes very succinctly. The result is a book whose chief merits, in comparison with other works on the same subject, are that it is compact and up to date. The succinct style adds pungency to passages devoted to discussion, but will make the mathematical passages seem hard to the student who tries to follow the details. Further, the author seems more at home on the formal side of mathematics than on the analytical. The conditions under which a theorem holds are sometimes sketchily stated or scattered through the preceding pages (see, for example, the theorem on p. 139), and some proofs taken from other authors have been slightly garbled in the process of condensation. Analytical terms are sometimes used in a slipshod way. For example, on p. 154 the author seeks the expectation $E(y/x)$ when $x$ and $y$ have a normal bivariate distribution. He seems unaware that $E(y/x)$ exists only as a Cauchy principal value: this invalidates his conclusion (p. 155) that the mean of $y/x$ in a sample of $n$ converges stochastically as $n\to\infty$. Moreover, having obtained by a formal calculation a series for $E(y/x)$ in powers of the coefficient of variation $v_1$ of $x$, he observes that it "converges only when ... $v_1$ is small": in fact it converges only if $v_1=0$, being, however, an asymptotic expansion for $E(y/x)$.

The book has a detailed table of contents, a good list of references at the end of each chapter, a number of examples, and an index; it is well printed. *H. P. Mulholland.*

***Laadi, Helene, and Boivie, O. Lagged product sums of H. Wold's normal deviates.** University Institute of Statistics, Uppsala, Sweden, 1952. xi+51 pp.

Using the tables given by H. Wold [Random normal deviates, Cambridge, 1948; these Rev. **10**, 553] the authors compute for each of the 500 columns of 50 random numbers, the lagged product sums:

$$S_k = x_1x_{k+1} + x_2x_{k+2} + \cdots + x_{50-k}x_{50}, \quad 0 \leqq k \leqq 26.$$

Using these tables, tests for the randomness of the deviates in Wold's tract are carried out. These tests involve the statistics $D = S_1/S_0$ and $D' = (S_1 + \alpha S_2 + \alpha^2 S_3 + \cdots)/S_0$, $\alpha = 1/4$. At the 2% critical level, these tests reject only 7 and 9 of the 500 columns respectively. *H. Chernoff.*

**Birnbaum, Z. W. Numerical tabulation of the distribution of Kolmogorov's statistic for finite sample size.** J. Amer. Statist. Assoc. **47**, 425–441 (1952).

If $F_N(x)$ is the sample cumulative distribution function for a sample of $N$ independent observations from a population with the continuous distribution function $F(x)$, $D_N = \sup |F(x) - F_N(x)|$ is Kolmogorov's statistic and has a distribution independent of $F$. A table is constructed for $P\{D_N < c/N\}$, $c = 1(1)15$, $N = 1(1)(100)$. A table is given for the 95 and 99 percentage points of $D_N$ for

$$N = 2(1)5(5)30(10)100.$$

*H. Chernoff* (Stanford, Calif.).

**Mainland, Donald, and Murray, I. M. Tables for use in fourfold contingency tests.** Science **116**, 591–594 (1952).

Tables are prepared to indicate whether the data of a four-fold contingency table is significant in testing the hypothesis of independence at the 5% and 1% critical levels. These tables apply for cases where the sums of the two rows are equal. The tables may, with interpolation, be used for row sums up to 500. It is indicated how these tables may sometimes suffice in cases of unequal row sums.

*H. Chernoff* (Stanford, Calif.).

**Béjar, Juan. Maxima and minima of the coefficients of asymmetry and kurtosis in finite populations.** Trabajos Estadística **3**, 3–11 (1952). (Spanish. French summary).

The upper bounds for $\beta_1$ and $\beta_2$ were given by Wilkins [Ann. Math. Statistics **15**, 333–335 (1944); these Rev. **6**, 91] and Picard [ibid. **22**, 480–482 (1951); these Rev. **13**, 141] respectively. An alternative development is provided and it is shown that

$$\min \beta_2 = \begin{cases} 1 & n \text{ even} \\ (n^2+3)/(n^2-1) & n \text{ odd} \end{cases}, \quad \min \beta_1 = -\max \beta_1.$$

*H. L. Seal* (New York, N. Y.).

**Lomnicki, Z. A. The standard error of Gini's mean difference.** Ann. Math. Statistics **23**, 635–637 (1952).

The author considerably simplifies the derivation of the variance of Gini's mean difference as compared to that originally given by U. S. Nair [Biometrika **28**, 428–436 (1936)]. He notes what appears to be a typographical error in Nair's general formula since Nair's results in special cases agree with those found by the use of Lomnicki's formula.

*C. C. Craig* (Ann Arbor, Mich.).

**Kupperman, Morton. On exact grouping corrections to moments and cumulants.** Biometrika **39**, 429–434 (1952).

**Barton, D. E., and Dennis, K. E. The conditions under which Gram-Charlier and Edgeworth curves are positive definite and unimodal.** Biometrika **39**, 425–427 (1952).

**Ogawa, Junjiro. Analytical derivation of sampling distribution of intraclass correlation coefficient.** Osaka Math. J. **4**, 69–76 (1952).

The author gives an analytical derivation of the sampling distribution of the ordinary correlation coefficient $r$ in the case of the bivariate normal distribution.

*R. P. Peterson* (Seattle, Wash.).

**Mulholland, H. P. On distributions for which the Hartley-Khamis solution of the moment-problem is exact.** Biometrika **38**, 74–89 (1951).

Investigation of those cumulative distribution functions (c.d.f.'s) for which the method of the title [Biometrika **34**, 340–351 (1947); these Rev. **9**, 623] for estimating a c.d.f. from its first $n$ moments, is exact. The method in question has a representation as a sum of multiples of derivatives of the $n$-fold convolution of rectangular distributions, analogous to that of the familiar Gram-Charlier Type A series (in terms of Hermite polynomials), examples being given where the first method is superior. Bounds are given on the error of the method. *J. Kiefer* (Ithaca, N. Y.).

**Kendall, M. G. Moment-statistics in samples from a finite population.** Biometrika **39**, 14–16 (1952).

L'auteur reprend le problème envisagé par Wishart dans Biometrika **39**, 1–13 (1952) [ces Rev. **14**, 296] et montre qu'on peut justifier la méthode suivie par Wishart par application d'idées que l'auteur a exposées dans Ann. Eugenics **10**, 106–111, 215–222, 392–402 (1940) [ces Rev. **1**, 347; **2**, 110, 235] et dans The advanced theory of statistics, v. I, 5ième éd., Griffin, London, 1952 [première éd. analysé dans ces Rev. **6**, 89]. *R. Fortet* (Paris).

**Murty, V. N. On a result of Birnbaum regarding the skewness of $X$ in a bivariate normal population.** J. Indian Soc. Agric. Statistics **4**, 85–87 (1952).

Let $(X, Y)$ have the bivariate normal distribution with zero expectations, unit variances, and correlation coefficient $\rho$, and let $\lambda(s)$ stand for $(e^{s^2/2}\int_s^\infty e^{-t^2/2}dt)^{-1}$. The reviewer has shown [Ann. Math. Statistics **21**, 272–279 (1950); these Rev. **11**, 673] that when this distribution is truncated to the set $Y \geqq \tau$, the third moment of $X$ becomes

$$E[X - E(X)]^3 = \rho^3\lambda(\tau)\{[\lambda(\tau) - \tau][2\lambda(\tau) - \tau] - 1\},$$

and conjectured that the quantity in braces is always $>0$. The author claims to give a proof of this conjecture but his argument shows only that the required inequality is true for sufficiently large positive $\tau$. *Z. W. Birnbaum.*

**Bates, Grace E., and Neyman, Jerzy. Contributions to the theory of accident proneness. I. An optimistic model of the correlation between light and severe accidents.** Univ. California Publ. Statist. **1**, 215–253 (1952).

Let $\Lambda$ be a positive random variable with distribution function $F(x)$, and let, for $\Lambda = \lambda$, $X_1, \cdots, X_s$ be $s$ independent Poisson-distributed random variables with means $a_1\lambda, \cdots, a_s\lambda$. Here $\Lambda$ represents the accident proneness of an individual drawn at random from a certain population,

while $X_1$ and $X_2, \cdots, X_s$ represent the number of severe accidents and the numbers of $s-1$ kinds of light accidents, respectively, incurred per unit time by the same individual. Various properties of the joint distribution of $X_1, \cdots, X_s$ are derived, especially for (*) $F'(x)=\{\beta^\alpha/\Gamma(\alpha)\}x^{\alpha-1}e^{-\beta x}$, in which case it is called the $s$-variate negative binomial distribution. E.g., for arbitrary $F(x)$ the conditional distribution of $X_1$, given $X_2, \cdots, X_s$, depends only on $Y=\sum_{i=2}^s X_i$, so that the problem of predicting severe accidents from light ones reduces to a two-dimensional problem. For the case when (*) holds and $a_1=1$ the authors obtain the maximum likelihood estimates of $\alpha$, $\beta$, and $A=\sum_{i=2}^s a_i$ based on $n$ independent observations on the pair $X_1$, $Y$, fit the bivariate negative binomial distribution of $X_1$ and $Y$ to actual data, and determine the conditional probability that $X_1=0$, given that a predetermined proportion of the mass of the $Y$-distribution is excluded by truncation. Finally they consider a scheme where the probability of surviving a severe accident is $\leqq 1$ (this is equivalent to the scheme ($N$) of Part II; cf. the following review). *D. M. Sandelius.*

**Bates, Grace E., and Neyman, Jerzy. Contributions to the theory of accident proneness. II. True or false contagion.** Univ. California Publ. Statist. 1, 255–275 (1952).

Considering an individual $I$ exposed to accidents of a specified kind, the authors introduce a generalization ($P$) of Pólya's contagious scheme, the main characteristics of ($P$) being that the probability $\theta$ of surviving an accident is $\leqq 1$, and that $\partial P_{m,\theta}(T_1, T_2)/\partial T_2|_{T_2=T_1}=-\lambda(1+\mu m)/(1+\nu T_1)$, where $\lambda$, $\mu$, $\nu$ are constants $\geqq 0$, and $P_{m,\theta}(T_1, T_2)$ denotes the probability that, during the time interval $(T_1, T_2)$, $I$ will incur no accidents, given that he incurred $m$ accidents during $(0, T_1)$ and lived at $T_1$. (In Pólya's original scheme $\mu=\nu$, $\theta=1$.) The authors compare ($P$) with the scheme ($N$) defined as ($P$) except that $\mu=\nu=0$ and $\lambda$ is a particular value of a gamma-distributed random variable. Their main result is that, except when $\lambda\mu=\nu$, ($P$) and ($N$) are distinguishable if certain accident data are available for at least two successive time intervals. *D. M. Sandelius* (Uppsala).

**Sukhatme, P. V., and Seth, G. R. Non-sampling errors in surveys.** J. Indian Soc. Agric. Statistics 4, 5–41 (1952).

Estimates of variance components are derived for various cases of the model $y_{ijk}^s=x_i^s+\alpha_j^s+\delta_{ij}^s+\epsilon_{ijk}^s$ ($i=1, \cdots, l_s$; $j=1, \cdots, m$; $k=1, \cdots, n_{ij}^s$; $s=1, \cdots, L$), where $y_{ijk}^s-x_i^s$ is the bias of $j^s$, the $j$th enumerator in the $s$th stratum, when observing $x_i^s$ for the $k$th time, $E[\epsilon_{ijk}^s|x_i^s, j^s]=0$, and $E[\delta_{ij}^s|j^s]=0$. A typical case is the one when the $mL$ enumerators are selected at random from a population of size $M$, and the $x$'s are allocated at random among the corresponding enumerators under the restrictions (i) $\sum_{j=1}^m n_{ij}^s=1$ or 2, (ii) the number of units observed once is the same for each enumerator, (iii) the number of units observed twice by the same enumerator is the same for all enumerators, and (iv) the number of units observed by two enumerators is the same for each pair of enumerators. Several numerical illustrations from actual surveys are given.

*D. M. Sandelius* (Uppsala).

**Sukhatme, P. V., and Narain, R. D. Sampling with replacement.** J. Indian Soc. Agric. Statistics 4, 42–49 (1952).

Variances of mean estimates and estimates of these variances are compared for two cases of two-stage sampling procedures, where $n$ p.s.u. (primary sampling unit(s)) are sampled with replacement and with probabilities proportionate to their sizes, namely, the case when sub-sampling of $m$ units from a p.s.u. is carried out independently every time it is selected, and the case when $m\lambda$ units are sub-sampled from a p.s.u. selected $\lambda$ times. Optimum values of $m$ and $n$ are derived under cost considerations. Finally, the variance of a mean estimate is derived for a third two-stage sampling procedure similar to the second one except that sampling goes on until $\nu$ distinct p.s.u. are obtained.

*D. M. Sandelius* (Uppsala).

**Midzuno, Hiroshi. On the sampling system with probability proportionate to sum of sizes.** Ann. Inst. Statist. Math., Tokyo 3, 99–107 (1952).

An unbiased mean estimator and its variance are obtained for a two-stage sampling procedure where $m$ p.s.u. (primary sampling units) are drawn without replacement, and the probability of selecting any set of p.s.u. is proportional to the sum of their sizes. The case $m=1$ was treated by Hansen and Hurwitz [Ann. Math. Statistics 14, 333–362 (1943); these Rev. 5, 210]. Two methods of obtaining the p.s.u. are described, one using varying probabilities at all $m$ steps, giving equal probabilities to all possible orders of selecting the p.s.u., and the other using equal probabilities at $m-1$ steps. (In the first formula on p. 103, $\binom{M-1}{M-p}$ should be replaced by $\binom{M-1}{m-1}$. The phrase "proportional to" should be inserted immediately before the second formula on p. 103.)

*D. M. Sandelius* (Uppsala).

**Ogawa, Junjiro. Contributions to the theory of systematic statistics. II. Large sample theoretical treatments of some problems arising from dosage and time mortality curve.** Osaka Math. J. 4, 41–67 (1952).

The author treats large sample problems of parameter estimation, hypothesis testing, and design of experiments connected with dosage mortality and time mortality curves by methods similar to those used in Part I [same J. 3, 175–213 (1951); these Rev. 13, 762]. *R. P. Peterson.*

**Goodman, Leo A. On the analysis of samples from $k$ lists.** Ann. Math. Statistics 23, 632–634 (1952).

The following problem is solved: given random samples from each of $k$ lists of names, estimate (a) the number of names occurring in common in pairs, triples, etc., of lists, (b) the number of names occurring in 1, 2, $\cdots$, $k$ lists. The unbiased estimators obtained are shown to be unique.

*D. G. Chapman* (Seattle, Wash.).

**Kitagawa, Tosio. Successive process of statistical inferences. II.** Mem. Fac. Sci. Kyūsyū Univ. A. 6, 55–95 (1951).

[For part I see same Mem. 5, 139–180 (1950); these Rev. 13, 854.] The author treats several cases involving so-called "two sample" and "three sample" theory. In the two sample theory, the first sample is used to obtain a prognostication of a function of the second sample (which need not be independent of the first sample). The three sample theory may be regarded as a generalization of the two sample theory where the prognostication is based on two previous samples. This theory introduces a new point of view when one regards the first sample as the previous knowledge existing before the immediate problem suggested the use of the second sample to prognosticate a function of the third sample. The author introduces a notion of relative efficiency of two kinds of prognostications in two-sample theory. These theories are also applied to problems involving finite populations. There are numerous typographical errors. *H. Chernoff.*

**Schmetterer, L. Über ein Beispiel aus der Statistik.** Z. Angew. Math. Mech. 32, 281–284 (1952).

Let $X_1, X_2, \cdots$ be independent normal $N(X_i|0, \sigma^2)$ and define $x^{(1)}=X_1+\cdots+X_n$, $x^{(2)}=X_1^2+\cdots+X_n^2$, $n$ being a chance variable independent of the $X_i$ having the geometric distribution $p^{n-1}q$, $p+q=1$. The paper investigates estimates of and confidence intervals for the parameter $q$ on the basis of $m$ observations on $x^{(1)}$ and $x^{(2)}$.

*G. E. Noether* (Boston, Mass.).

**Woodruff, Ralph S. Confidence intervals for medians and other position measures.** J. Amer. Statist. Assoc. 47, 635–646 (1952).

**Terpstra, T. J. A confidence interval for the probability that a normally distributed variable exceeds a given value, based on the mean and the mean range of a number of samples.** Appl. Sci. Research A. 3, 297–307 (1952).

The author modifies the procedure of Johnson and Welch [Biometrika 31, 362–389 (1940); these Rev. 1, 346] for obtaining a confidence interval for the probability that a normally distributed variable exceeds a given value, by replacing the sample standard deviation by a multiple of the mean range of several subsamples, in the calculation of a non-central $t$. This use of the mean range is justified by the results of Patnaik [Biometrika 37, 78–87 (1950); these Rev. 12, 116]. The two procedures are compared empirically by sampling experiments. Four nomographs are given for finding the confidence intervals directly from non-central $t$ ratios involving the mean range.

*S. W. Nash.*

**Moore, P. G. The estimation of the Poisson parameter from a truncated distribution.** Biometrika 39, 247–251 (1952).

**Cavé, René. Contrôle statistique par calibres modifiés d'efficacité optima.** C. R. Acad. Sci. Paris 235, 935–937 (1952).

**Rijkoort, P. J. A generalisation of Wilcoxon's test.** Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 394–404 (1952).

The author generalises Wilcoxon's test to the case of $k$ samples. Approximations are derived for the frequency function of the test statistic, and are compared with the exact frequency function for some special cases.

*P. Whittle* (Uppsala).

**Kruskal, William H. A nonparametric test for the several sample problem.** Ann. Math. Statistics 23, 525–540 (1952).

Given $C$ independent samples of sizes $n_1, \cdots, n_C$ from univariate populations with continuous cdf's $F_1, \cdots, F_C$. The paper uses the standard one-way analysis of variance $F$-ratio applied to the ranks of the observations in the overall sample of size $n_1+\cdots+n_C$ to test the hypothesis that $F_1=\cdots=F_C$. With proper normalization the statistic is shown to have an asymptotic chi-square distribution with $C-1$ degrees of freedom under the null hypothesis provided certain general conditions are satisfied. A necessary and sufficient condition for the consistency of the test with respect to a given alternative is given. A difference equation is derived for the small-sample distribution of the statistic under the null hypothesis. Possible extensions to discontinuous distributions are considered.

*G. E. Noether.*

**Massey, F. J., Jr. Correction to "A note on the power of a nonparametric test".** Ann. Math. Statistics 23, 637–638 (1952).

See same Ann. 21, 440–443 (1950); these Rev. 12, 117.

**Matthai, Abraham, and Kannan, M. B. The applicability of large sample tests for moving average and autoregressive schemes to series of short length—an experimental study. Part 1: Moving averages.** Sankhyā 11, 218–238 (1951).

The authors construct twenty-five 35-term series and fifty 15-term series for each of the three moving average schemes:

(A) $\xi_t=\zeta_t+\zeta_{t-1}$;

(B) $\xi_t=\zeta_t+\frac{1}{2}\zeta_{t-1}$;

(C) $\xi_t=\zeta_t+\zeta_{t-1}+\zeta_{t-2}+\zeta_{t-3}$.

They study Wold's test for such series. The test is found to be better after correction for bias due to the smallness of the samples, but still not very reliable. They also investigate the problem of fitting the parameters of such series, with good results. In addition, they try fitting a two-parameter autoregressive scheme to three-parameter series of type (C), with poor results.

*A. Blake* (Buffalo, N. Y.).

**Rao, S. Raja, and Som, Ranjan K. The applicability of large sample tests for moving average and autoregressive schemes to series of short length—an experimental study. Part 2: Autoregressive series.** Sankhyā 11, 239–256 (1951).

The authors construct twenty-five 35-term series and fifty 15-term series for each of the two autoregressive schemes

(I) $\xi_t-0.8\xi_{t-1}=\zeta_t$;

(II) $\xi_t-0.7\xi_{t-1}+0.6125\xi_{t-2}=\zeta_t$.

They study the bias due to the smallness of the samples in the serial correlation coefficients for these series, and find it to be considerable for model (I) but small for model (II). They study the applicability of Quenouille's test for the goodness of fit of given models of these types with given coefficients. The test is found to give many spurious indications of significance. Some reduction in this anomaly is achieved by correcting for bias. The authors consider also the problem of fitting the coefficients in model (II) by three methods, viz., by equating population correlations to the corresponding observed serial correlations, and by minimizing certain two quadratic forms in residuals; quite large errors occur in these fits. They also try fitting a three-parameter moving average to the two-parameter scheme (II), obtaining good fits as judged by Wold's test.

*A. Blake* (Buffalo, N. Y.).

**Rao, C. Radhakrishna. The applicability of large sample tests for moving average and autoregressive schemes to series of short length—an experimental study. Part 3: The discriminant function approach in the classification of time series (Part III of Statistical inference applied to classificatory problems).** Sankhyā 11, 257–272 (1951).

For the purpose of fitting $n$ parameters in a moving average or autoregressive scheme, it is not efficient to use estimates based on only the first $n$ observed serial correlations. The author studies this point quantitatively, in part by means of the series constructed as described in the preceding two reviews, and proposes a sequential method by which to save the labor of dealing with unnecessarily many serial correlation coefficients.

*A. Blake* (Buffalo, N. Y.).

**Patterson, H. D. The construction of balanced designs for experiments involving sequences of treatments.** Biometrika 39, 32–48 (1952).

The author considers experimental designs in which treatments are applied to the experimental material in a number of successive periods. In order that the treatment effects and the errors can be estimated by a reasonably simple statistical analysis certain elements of balance in the design are desirable (depending on what residual effects are to be estimated). For estimating the direct and first residual effects only, the construction of the designs involves the solution of the following combinatorial problem. It is required to place $v$ symbols $1, 2, \cdots, v$ (denoting treatments) in the cells of a rectangular scheme with $k$ rows (periods) and $b$ columns (sequences) satisfying the following conditions. (I) No treatment symbol occurs in a given sequence more than once. (II) Each symbol occurs in a period an equal number of times. (III) Every two treatment symbols occur together in the same number $\lambda$ of sequences. (IV) Each ordered succession of two treatment symbols occurs equally often in the sequences. (V) Every two treatment symbols occur together in the same number of curtailed sequences formed by omitting the final period. (VI) In those sequences in which a given treatment occurs in the final period the other treatments occur equally often. (VII) In those sequences in which a given treatment occurs in any but the final period, each other treatment occurs equally enough in the final period. Conditions I and III ensure that the columns give blocks of balanced incomplete block designs [Yates, Ann. Eugenics 7, 121–140 (1936); Bose, ibid. 9, 353–399 (1939)]. Condition II gives double balance for simple estimation of period effects [Shrikhande, Ann. Math. Statistics 22, 235–247 (1951); these Rev. 12, 843; and Hartley, Biometrics 7, 117–118 (1951)]. The conditions IV–VII are necessary for simple estimation of residual effects and further restrict the class of available designs. It is necessary that $b/v$ be an integer and that $b(k-1)/v \equiv 0 \bmod (v-1)$. The construction of these designs is discussed in general terms and application is then made to the following special cases (1) $v=k$, $b=v$ or $2v$ according as $v$ is even or odd, (2) $v=$a prime power, $k\leqq v$, $b=v(v-1)$, (3) $v=$a prime of the form $4n+3$, $k=3$, $b=v(v-1)/2$, (4) $v=$a prime of the form $4n+3$, $k=(v+1)/2$, $b=2v$, (5) $v=$a prime of the form $4n+3$, $k$ odd, $b=v(v-1)/2$. A number of other designs are also obtained, and two of the designs with $k=3$ are found to be non-existent. Actual solutions have been sought for all cases in the range $3\leqq k\leqq 6$, $b\leqq 60$. *R. C. Bose.*

## *Mathematical Economics*

**Bellman, Richard. On the theory of dynamic programming.** Proc. Nat. Acad. Sci. U. S. A. 38, 716–719 (1952).

Statement without proofs of existence and uniqueness theorems, and in some cases forms of the solution, of various instances of the functional equation $f(p)=\min_k T_k(f)$ where $T_k$ is an operator and the index $k$ ranges over a suitable space. Such equations arise in problems of minimizing some total expected measure of loss by choosing suitable actions at each stage of observation of a stationary (in some sense) stochastic process; for example, problems of the type considered by Dvoretzky, the reviewer, and Wolfowitz [Econometrica 20, 187–222 (1952); these Rev. 13, 856]. Some of the results (e.g., Theorems 1 and 6 respectively) of the present paper may be obtained without difficulty from the methods of the above reference and of Wald and Wolfowitz [Proc. Nat. Acad. Sci. U. S. A. 35, 99–102 (1949); Ann. Math. Statistics 21, 82–99 (1950); these Rev. 10, 466; 11, 529]. *J. Kiefer* (Ithaca, N. Y.).

**May, Kenneth O. A set of independent necessary and sufficient conditions for simple majority decision.** Econometrica 20, 680–684 (1952).

A group of $n$ individuals is to choose between a pair of alternatives $x$ and $y$. The problem consists in prescribing a method or function which will attach to any set of preferences of the individuals a preference of the group. Such a function is called a "group decision function". The decision function is required to satisfy the following four conditions which we state roughly: (1) the function is to be single-valued and defined for all sets of individual preferences; (2) it is to be symmetric with respect to the $n$ individuals; (3) it is to be symmetric with respect to the alternatives $x$ and $y$; (4) a condition making the decision function responsive to the preferences of the individuals. Theorem: A decision function reduces to the method of simple majority decision if and only if it satisfies conditions (1)–(4). The four conditions are shown to be consistent and independent. *D. Gale* (Providence, R. I.).

**Cherubino, Salvatore. Sulla matrice-moltiplicatore dei settori economici.** Ann. Mat. Pura Appl. (4) 33, 247–254 (1952).

Goodwin [Economic J. 59, 537–555 (1949)], Chipman [Econometrica 18, 355–374 (1950); these Rev. 13, 369] and others have considered statical and dynamical many-sector multiplier systems in which the matrix of marginal propensities to spend is usually thought of as having non-negative constants as elements. The author calls attention to the possibility of a more general formulation where the constant marginal propensities to spend are replaced by analytic functions of a matrix $X$ of prices. This leads to equations $dY=f(X)dE$, where $Y$ and $E$ are vectors of sector incomes and autonomous expenditures and $f(X)$ is a matrix of functions of $X$. The matrix $f$ is assumed to belong to a complex algebra $A$ and $X$ to a set of nonsingular matrices contained in the center of $A$. These assumptions seem to have no particular economic meaning. In addition the injections $E$ are permitted to be functions of $X$ and $Y$ and then specialized so that the differential equations read $dY=F(X)YdX$. Some properties of a solution in terms of a generalized Volterra integral are mentioned. *R. Solow.*

**Wold, H. Ordinal preferences or cardinal utility?** (With additional notes by G. L. S. Shackle, L. J. Savage, and H. Wold.) Econometrica 20, 661–664 (1952).

Recently von Neumann and Morgenstern [Theory of games and economic behavior, 2nd ed., Princeton Univ. Press, 1947, p. 26; these Rev. 9, 50], Marschak [Econometrica 18, 111–141 (1950); these Rev. 12, 515], Savage and others have proposed systems of axioms defining rational behavior in risky situations from which can be deduced the existence of a conceptually measurable utility function for each rational individual with the property that of any pair of probability-income situations he always prefers the one with the higher expected utility. Let $A$, $B$, and $C$ be generalized lottery tickets specifying various prizes (which may include lottery tickets) with specified probabilities, and let $(A, B)p$ represent the lottery ticket which yields $A$ with probability $p$ and $B$ with probability $1-p$.

Then each system of axioms contains one which states, in one form or another, that if a rational man is indifferent between $A$ and $B$, and if $C$ and $p$ are arbitrary, he is indifferent between $(A,C)p$ and $(B,C)p$. Wold argues that if choices are thought of as occurring in time, and if commodities are storable, this "independence axiom" may be extremely artificial. Savage replies that the theory of choice in risky situations as developed is not meant to cover this sort of situation, and in particular once-for-all choices need imply nothing about choices when the same alternatives are presented in a repetitive fashion. *R. Solow.*

**Manne, Alan S. The strong independence assumption—gasoline blends and probability mixtures.** (With additional note by A. Charnes.) Econometrica **20**, 665–669 (1952).

Charnes, Cooper and Mellon [Econometrica **20**, 135–159 (1952)] have used linear programming methods to find the optimal blending program for aviation gasolines in an oil refinery. The assumptions of linear programming involve what is essentially the independence axiom of the previous review. (Let $A$, $B$, and $C$ be gasoline stocks; $(A,B)p$ be the octane or performance rating of a gasoline obtained by blending $A$ and $B$ in proportions $p:1-p$; and instead of "be indifferent to" read "have the same octane rating as".) The author notes this equivalence and points out that the usual formulas for blending gasolines do not satisfy the independence axiom. Charnes, in reply, states that in the original work the independence assumption was assumed to hold locally only, and that the results were checked against refinery experience. *R. Solow* (Cambridge, Mass.).

**Samuelson, Paul A. Probability, utility, and the independence axiom.** Econometrica **20**, 670–678 (1952).

The author sketches a set of axioms which together imply the existence of a utility function $u(X)$, unique up to an increasing linear transformation, defined for points in a space of commodities, such that when presented with a choice among risky situations an individual behaving according to the axioms will act as if he is maximizing $Eu=\sum p_i u(X_i)$, where each chance involves the receipt of one and only one prize $X_1$ or $X_2, \cdots$, with probabilities $p_1, p_2, \cdots$, respectively.

The axioms are: 1. The individual has a complete preference ordering of all risky situations (generalized lottery tickets). 2. If a lottery ticket contains other lottery tickets as prizes, and so forth, these may be reduced according to the usual rules of probability, to yield a lottery ticket equivalent in the preference ordering. In the notation of the preceding two reviews, $(A,(A,B)p_1)p_2$ and $(A,B)p_2+(1-p_2)p_1$ are indifferent choices. 3. The independence axiom stated above, with $p=\frac{1}{2}$. 4. Certain closure or continuity assumptions. The result is closely related to a theorem of Nagumo, Kolmogoroff, and de Finetti [Hardy, Littlewood, and Pólya, Inequalities, 1st ed., Cambridge Univ. Press, 1934, Theorem 215, p. 158].

The plausibility of the axioms is discussed. It is pointed out that the stochastic independence axiom implies nothing about the dependence or independence of preferences for sure collections of commodities. The case of repetitive choice over time is to be handled by specifying complete programs of consumption and complete strategies for sequences of choices. See the preceding two reviews. *R. Solow.*

**Malinvaud, E. Note on von Neumann-Morgenstern's strong independence axiom.** Econometrica **20**, 679 (1952).

The von Neumann-Morgenstern axioms for measurable utility seem to contain explicitly nothing to correspond to the independence axiom mentioned in the above reviews. This note verifies Samuelson's conjecture that this axiom is actually present in implicit form. Von Neumann and Morgenstern define an operation which may be written in the notation of the previous reviews: $(A,B)p=C$ where now $A, B, C$ represent equivalence classes induced by the relation of indifference. It is easily seen, by taking two distinct members of $A$ and the same member of $B$, that this operation makes sense, that is, that $C$ is indeed an equivalence class if and only if the independence axiom holds.

*R. Solow* (Cambridge, Mass.).

## *Mathematical Biology*

**Rashevsky, N. Mathematical biology of division of labor between two individuals or two social groups.** Bull. Math. Biophys. **14**, 213–227 (1952).

The method of maximising satisfaction functions is applied to a situation in which two individuals may each produce two needed objects of satisfaction, or may each produce only one of the objects ("division of labor") and then make a partial exchange. Under certain assumptions, including a logarithmic form of the satisfaction function, it is found that there is no advantage in division of labor, unless such division materially increases the purely physical efficiency of production; but it is made plausible that satisfaction functions which have an asymptote could well lead to different conclusions. The case is also studied in which division of labor occurs between two groups of individuals and their relative sizes are determined from considerations of maximum satisfaction. Possible applications to problems of urbanization are suggested. *I. M. H. Etherington.*

**Armitage, P. The statistical theory of bacterial populations subject to mutation.** J. Roy. Statist. Soc. Ser. B. **14**, 1–33; discussion, 34–40 (1952).

"This paper is a study of the growth of a mixed bacterial population composed of two types of organisms, and subject to mutation from one type to the other, in one or both directions. The bacteria divide by binary fission, but the rates of division for the two types may differ. The paper is intended as a unified and critical presentation of statistical results previously published by other authors, together with a number of new results.

"The first part presents a mathematical theory, from both the deterministic and the stochastic viewpoints. The definition of a mutation rate is discussed in some detail. Particular attention is directed to the situation in which the proportion of organisms of one type remains negligible throughout the period of growth (so-called short-term experiments); the effects of delayed phenotypic expression and of the sampling of replicate cultures are considered. For the more general situations, approximate expressions are obtained for the first few cumulants of the joint distribution of the numbers of organisms of each type present at any time.

"The second part is concerned with methods of estimation of the parameters of a mutation process. In particular, I examine the suitability of various methods of estimating the

mutation rate in short-term experiments, when phenotypic delay is present. Methods based on the upper portion of the distribution of the number of mutants in replicate cultures, such as the upper quartile, are preferable.

"The verification of the theory by certain data published by other authors is considered in the third part. An attempt is made to fit some observed distributions of the number of mutants in replicate cultures, using a particular model for phenotypic delay; the fit is unsatisfactory, the fitted distributions having too high a frequency of cultures with two or three mutants." (Author's summary.)

Much of the interest in bacteriological variations recently has been centered around the question whether they appear by mutation or are induced by adaptation to environment. Although statistical studies of Luria and Delbrück [Genetics **28**, 491–511 (1943)] seemed to support the former hypothesis, the present paper and the discussion following it indicate that the question is still controversial.

*T. E. Harris* (Santa Monica, Calif.).

**Kendall, David G. On the choice of a mathematical model to represent normal bacterial growth.** J. Roy. Statist. Soc. Ser. B. **14**, 41–44 (1952).

"The problem considered is that of constructing a stochastic process (not necessarily of the Markov type) to represent the growth of a bacterial colony in the absence of mortality and mutation. The importance of the distribution of fission-times is emphasized, and an approximate relation between the coefficient of variation of the fission-times and the asymptotic coefficient of variation of clone-size is discussed. Rahn's model of the fission process, a rather different model put forward by the author in 1948 and a more general model (of which the other two are special cases) are examined in relation to the problem of estimating "the number of genes" in bacteria. Reasons are given for supposing that existing estimates of this number (Finney and Martin, 25; Kendall, 20) may be too low. The need for further experimental work (including an experimental determination of clone-size fluctuations) is stressed." (Author's summary.) *T. E. Harris* (Santa Monica, Calif.).

**Malécot, G. Un traitement stochastique des problèmes linéaires (mutation, linkage, migration) en génétique de population.** Ann. Univ. Lyon. Sect. A. (3) **14**, 79–117 (1951).

Continuing his work on genetics [Ann. Univ. Lyon. Sect. A. (3) **13**, 37–60 (1950); these Rev. **13**, 263], the author attempts to generalize the previous discussions. Consider, in a population consisting of a number of restricted groups, two gametes drawn at random, the one from the $i$th group and the other from the $j$th group, at the beginning of the $n$th generation. The probability, estimated at the initial date, of the event that the homologous loci of these two gametes are the carriers of the identical genes, namely the genes originating in a common locus of an ancester, is denoted by $f_n(i, j)$, $f_n(i, j]$, $f_n[i, j)$ or $f_n[i, j]$, where the left and right parentheses or brackets relate to male or female gametes of the $i$th and $j$th groups, respectively. In cases of autosomal genes and of genes carried by the sex chromosomes, the author derives the recurrence relations satisfied by these "coefficients of mean parentage", which, being able to be arranged appropriately in a matrix form, are of the character of difference equations.

The main task is to investigate the evolution of the population, especially the asymptotic behaviors, as $n\to\infty$, of the solutions of the fundamental equations thus obtained, by taking the mutation into account and by supposing that the effectives of the groups and the rates of migration are known and further that the various mating systems are designated; special attention is paid to the case of random mating, and to the case of total homogamy in selective mating. The investigations of the effects of linkage are also considered, taking into account the phenomenon of crossing-over together with mutation. The illustrations are given in detail, especially for cases of isolated groups, for cases of one-dimensional homogeneous migration, and also for cases where the number of groups is few or extremely large.

*Y. Komatu* (Tokyo).

**Marchand, Henri. Analogie entre une loi d'union sélective et une loi de fécondité ou de survivance différentielles.** C. R. Acad. Sci. Paris **235**, 863–864 (1952).

# TOPOLOGY

**Dirac, G. A. Connectivity theorems for graphs.** Quart. J. Math., Oxford Ser. (2) **3**, 171–174 (1952).

If $S_1$ and $S_2$ are disjoint non-null sets of vertices of a graph $G$, let a path in $G$ from a member of $S_1$ to a member of $S_2$ which has no intermediate vertex in $S_1$ or $S_2$ be called an $S_1S_2$-way. The author proves the following theorem: If $S_1$ and $S_2$ are not separated by a vertex or edge and each contains at least two vertices and no cut-vertex, and if $W$ is any given $S_1S_2$-way, then the graph contains two completely disjoint $S_1S_2$-ways in each of which the vertices which belong to $W$, if any, occur in the same order as in $W$.

*W. T. Tutte* (Toronto, Ont.).

**Dirac, G. A. Map-colour theorems.** Canadian J. Math. **4**, 480–490 (1952).

A map drawn on a surface is called $k$-chromatic if the countries of the map can be coloured (in the sense of the four-colour problem for the sphere) with $k$ colours, but not with $k-1$ colours. $k$ is called the chromatic number of the map. Heawood proved [Quart. J. Pure Appl. Math. **24**, 332–338 (1890)] that the chromatic number of a map on a surface of connectivity $h>1$ is at most $n_h$, where $n_h=[\frac{1}{2}\{7+(24h-23)^{1/2}\}]$; $h$ is the constant in Euler's polyhedral formula $N+F-E=3-h$. In many cases $n_h$-chromatic maps exist, as it is possible to draw maps consisting of $n_h$ mutually adjacent countries on the surfaces in question. (For $2\leqq h\leqq 15$ the complete $n_h$-graph can be drawn on at least one surface of connectivity $h$.) The author proves the following theorem for $h=3$ and $h\geqq 5$: If on a surface of connectivity $h$ an $n_h$-chromatic map can be drawn, then that map contains a set of $n_h$ mutually adjacent countries. For $h=2$ or 4 the question is undecided, but the author proves the following weaker result in these cases: any $n_h$-chromatic map can be reduced to a set of $n_h$ mutually adjacent countries once the right of annexation is proclaimed.

The paper includes a simple proof of Heawood's result quoted above, and one for Heawood's 5-colour-theorem for the sphere. *N. G. de Bruijn* (Amsterdam).

★**Sierpinski, Waclaw. General topology.** Translated by C. Cecilia Krieger. Mathematical Expositions, No. 7, University of Toronto Press, Toronto, 1952. xii+290 pp. $6.00.

This book is a revision of the author's earlier "Introduction to general topology" [Univ. of Toronto Press, 1934],

and, for convenience, we describe it by comparing it with the earlier version. The author begins with a Frechet (V) space, which is a set together with a correspondence associating with each point a family of sets called neighborhoods. Making no assumptions on the neighborhoods, he proceeds via the notion of derived set to closed sets and open sets, and it turns out that the open sets so defined satisfy the axioms for a topological space except that the intersection of two open sets may fail to be open. With these weak assumptions a number of theorems on continuity, separation, connectedness and compactness are derived. These extend considerably the results given in the first chapter of the earlier book. In Chapter 2 axioms for a topological space are assumed: more precisely, the neighborhood system is further restricted so that the neighborhoods are open, the intersection of two neighborhoods contains a neighborhood, and the Kolmogoroff axiom holds (in the nomenclature of Alexandroff and Hopf, "Topologie" [Springer, Berlin, 1935], a $T_0$ space). After spaces whose topologies have a countable base (sometimes called perfectly separable) are considered, Hausdorff spaces which satisfy the first axiom of countability are studied—this represents a considerable change from the earlier work. The remainder of the book covers roughly the same material as the original, but with some deletions and additions and many improvements. A welcome and important improvement is the collection of problems and examples which has been added. These are amusing and instructive. There are many changes in terminology, representing for the most part usage which has come to be more or less standard since the first edition. The writing displays Sierpinski's usual lucidity.

C. C. Krieger's appendix on cardinal and ordinal numbers, which appeared in the earlier volume, has been reproduced almost unchanged.

Although minor faults can be found, the most serious criticism of the book lies in the topics omitted rather than in the material covered. The treatment of compactness seems incomplete; bicompactness is defined by "each infinite set has an accumulation point of the same cardinal" and it is not proved that this is equivalent to the usual covering definition. There is essentially no discussion of product spaces or quotient spaces, very little of function spaces, and no mention of uniform spaces or of other extensions of the notion of metric space. *J. L. Kelley.*

**Talmanov, A. D. On extension of continuous mappings of topological spaces.** Mat. Sbornik N.S. **31**(73), 459–463 (1952). (Russian)

Let $S$ be a $T_1$-space, $A$ a dense subspace of $S$, and $R$ a compact Hausdorff space. Let $f$ be a continuous mapping of $A$ into $R$. Then $f$ admits a continuous extension over $S$ if and only if for all disjoint closed subsets $A_1$, $A_2$ of $R$, the relation $(f^{-1}(A_1))^- \cap (f^{-1}(A_2))^- = 0$ obtains (closure in $S$). From this result, a theorem of Yu. M. Smirnov [Uspehi Matem. Nauk **6**, no. 4(44), 204–206 (1951)] is easily proved, as well as a theorem of Vulih [Mat. Sbornik N.S. **30**(72), 167–170 (1952); these Rev. **14**, 70]. A final corollary is a special case of a theorem widely known and recently published by Katětov [Fund. Math. **38**, 85–91 (1951); these Rev. **14**, 304]. *E. Hewitt* (Seattle, Wash.).

**Shirota, Taira. A class of topological spaces.** Osaka Math. J. **4**, 23–40 (1952).

Let $X$ be a completely regular space; let $eX$ be the uniform structure on $X$ whose basis is the family of all countable normal coverings of $X$ [Tukey, Convergence and uniformity in topology, Princeton, 1940, pp. 43 ff.; these Rev. **2**, 67.] First showing that $eX$ induces a topology compatible with that of $X$, the author proves that the following conditions are equivalent: (A) $eX$ is a complete uniform structure; (B) $X$ is a $Q$-space as defined by the reviewer [Trans. Amer. Math. Soc. **64**, 45–99 (1948); these Rev. **10**, 126]; (C) $X$ is homeomorphic to a closed subset of a Cartesian product of lines [see in this connection the reviewer's paper loc. cit., Theorem 60]. Questions of completeness are next discussed, the principal new result being that a space $X$ whose character is weakly accessible from $\aleph_0$ and which admits a complete uniform structure is $e$-complete. The reviewer's Theorem 56 [loc. cit.] is proved by completing the space $X$ under the uniform structure $eX$. This theorem has also been proved by Nachbin, who has considered the coarsest uniform structure on $X$ under which all continuous real-valued functions are uniformly continuous. This structure is obviously isomorphic to the structure $eX$. Other results on subsets, continuous images, and the like, are also obtained.

The final sections take up relations between the topology of a $Q$-space $X$ and the algebraic structure of the set $\mathfrak{C}(X, R)$ of all continuous real-valued functions on $X$. Here $\mathfrak{C}(X, R)$ is considered as a translation lattice in Kaplansky's sense [Amer. J. Math. **70**, 626–634 (1948); these Rev. **10**, 127]. For $Q$-spaces $X$ and $Y$, one finds that $X$ and $Y$ are homeomorphs if and only if $\mathfrak{C}(X, R)$ and $\mathfrak{C}(Y, R)$ are isomorphic as translation lattices; the proof is similar to Kaplansky's [loc. cit.]. In like manner, a $Q$-space $X$ is completely characterized up to homeomorphisms by $\mathfrak{C}(X, R)$ regarded as a lattice-ordered group. A characterization is also given of vector lattices which are imbeddable in $\mathfrak{C}(X, R)$ in a certain way. *E. Hewitt* (Seattle, Wash.).

**Roberts, G. T. Topologies in vector lattices.** Proc. Cambridge Philos. Soc. **48**, 533–546 (1952).

Let $X$ and $Y$ be two real vector spaces and $\langle x, y\rangle$ a real functional on $X \times Y$ which is linear and total in $x \in X$ and $y \in Y$ separately. If $P$ is a subset of one of the spaces, say of $X$, let its polar $P^0$ be the subset of the other space, namely $Y$, of all $y \in Y$ with $|\langle x, y\rangle| \leqq 1$ for $x \in P$. Let $\mathcal{P}$ be a collection of subsets of say $X$ with: (i) if $P_1, P_2 \in \mathcal{P}$, there is a $P_3 \in \mathcal{P}$, $P_1 \cup P_2 \subset P_3$; (ii) $\bigcup \mathcal{P} = X$; (iii) if $P_1 \in \mathcal{P}$ and $\lambda$ is real, there is a $P_2 \in \mathcal{P}$, $\lambda P_1 \subset P_2$; (iv) $\sup \{|\langle x, y\rangle|; x \in P\} < \infty$ for $y \in Y$ and $P \in \mathcal{P}$. Then $\mathcal{P}$ is said to define a boundedness in the pair $X$, $Y$, and $\mathcal{P}$ determines a topology $t$ making $Y$ into a topological vector space, a basic neighborhood of 0 in $Y$ being $P^0$, $P \in \mathcal{P}$. Similarly, the family $\mathfrak{Q}$ of $t$-bounded subsets of $Y$ defines a boundedness in the pair $Y$, $X$ and so a topology in $X$. Usually one takes $\mathcal{P}$ on $X$ as the collection of all finite subsets of $X$ (weak topology defined by $X$ on $Y$) or as the collection of all convex subsets of $X$ compact in the weak topology defined by $Y$ on $X$ (Arens-Mackey topology defined by $X$ on $Y$). In this paper, the author considers a vector lattice $K$, the complete vector lattice $K^c$ of all order-bounded linear functionals on $K$ with $\langle x, y\rangle = y(x)$, $x \in K$, $y \in K^c$ and, besides the two usual choices of $\mathcal{P}$ in $K$ or in $K^c$, he takes also the collection of all order-bounded subsets of $K$ or $K^c$. He then obtains several topologies on $K$ and $K^c$ and discusses them in general and also in a few special cases ($K$ a complete vector lattice, etc). Some of the author's results bear some relationship to the reviewer's note in Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. 1, pp. 464–465 [Amer. Math. Soc., Providence, R. I., 1952]. *L. Nachbin* (Rio de Janeiro).

**Dowker, C. H. On a theorem of Hanner.** Ark. Mat. 2, 307–313 (1952).

Let $X$ be a Hausdorff space. $X$ has the extension property relative to a topological space $Y$ if every continuous function from a closed subset of $X$ into $Y$ can be extended to a continuous function from $X$ into $Y$. It is known, by virtue of a result of Hanner [Ark. Mat. 1, 375–382 (1951); these Rev. 13, 266] and a result of Dugundji [Pacific J. Math. 1, 353–367 (1951); these Rev. 13, 373], that $X$ has the extension property relative to all separable Banach spaces if and only if $X$ is normal. The author's principal result essentially asserts that, analogously, $X$ has the extension property relative to all Banach spaces if and only if $X$ is collectionwise normal in the sense of Bing [Canadian J. Math. 3, 175–186 (1951); these Rev. 13, 264]. This strengthens a result of Arens [Pacific J. Math. 2, 11–22 (1952); these Rev. 14, 191], who proved that if $X$ is fully normal, then $X$ has the extension property relative to all Banach spaces.

*E. Michael* (Chicago, Ill.).

**Hanner, Olof. Retraction and extension of mappings of metric and non-metric spaces.** Ark. Mat. 2, 315–360 (1952).

Let $Q$ be a class of topological spaces, and let $X$ be a topological space. $X$ is an AR($Q$) (resp. ANR($Q$)) if it is a $Q$-space, and if, whenever $X$ is a closed subset of a $Q$-space $Y$, then $X$ is a retract of $Y$ (resp. some neighborhood of $X$ in $Y$). $X$ is an ES($Q$) (resp. NES($Q$)) if, whenever $Y$ is a $Q$-space, then any continuous function from a closed subset $B$ of $Y$ into $X$ can be extended to a continuous function from $Y$ (resp. some neighborhood of $B$ in $Y$) into $X$. The space $X$ is a local NES($Q$) if each point of $X$ has a neighborhood which is an NES($Q$). In this paper, the author continues the study of these concepts which he began in the paper referred to in the previous review and in Ark. Mat. 1, 389–408 (1951) [these Rev. 13, 266]. The following is a summary of the major results. (1) Let $Q$ be any of the following classes of spaces: normal, collectionwise normal [cf. the previous review], fully normal (=paracompact), Lindelöf (i.e., regular, and every open covering has a countable subcovering), compact, metric, separable metric, and compact metric. Then a $Q$-space is an AR($Q$) (resp. ANR($Q$)) if and only if it is an ES($Q$) (resp. NES($Q$)). (2) Let $X$ be an ANR(metric); then (a) $X$ is an ANR (fully normal)$\rightleftarrows X$ is an ANR(collectionwise normal)$\rightleftarrows X$ is an absolute $G_\delta$ (b) $X$ is an ANR(normal)$\rightleftarrows X$ is a separable absolute $G_\delta$ (c) $X$ is an ANR(completely regular)$\rightleftarrows X$ is separable and locally compact. These results remain true if "ANR" is replaced by "AR", except that in (c) "separable and locally compact" must be replaced by "compact". (3) Let all $Q$-spaces be (fully normal/collectionwise normal/normal); then any (arbitrary/fully normal/Lindelöf) local NES($Q$) is an NES($Q$). (4) Every (not necessarily finite or locally finite) simplicial complex, with either the metric topology or J. H. C. Whitehead's "weak" topology, is an NES(metric). The paper ends with a number of homotopy theorems, which the author had previously proved for separable metric spaces, and which are stated here for more general spaces. One of these theorems answers affirmatively a question raised by J. Dugundji at the end of the paper referred to in the previous review. [Reviewer's note: The part of (4) dealing with the "weak" topology has been obtained independently by J. Dugundji [Portugaliae Math. 11, 7–10 (1952); these Rev. 14, 74]].

*E. Michael.*

**Hashimoto, Hiroshi. On some local properties on spaces.** Math. Japonicae 2, 127–134 (1952).

Consider a property P of subsets $X$ of a given topological space; for any $X$, let $X^*$ denote the set of points at which $X$ has not the property P [cf., e.g., Kuratowski, Topologie I, 2nd ed., Warszawa-Wroclaw, 1948, p. 34; these Rev. 10, 389]. Supposing that (1) P is hereditary, (2) P is countably additive, (3) if $XX^*=0$, then $X$ has property P, (4) $X^*\subset \operatorname{Int} \bar{X}$, the author investigates properties of $X^*$ as well as of the topology which is obtained by putting the closure of $X$ equal to $X+X^*$.

*M. Katětov* (Prague).

**McCandless, Byron H. Dimension and disconnection.** Proc. Amer. Math. Soc. 3, 657–658 (1952).

The author proves that, for a semi-compact separable metric space $X$, $\dim X\leqq n$ if and only if any closed subset of $X$ containing at least two points can be disconnected by a closed set of dimension less than or equal to $n-1$. Presumably, semi-compact means that $X$ is a countable union of compact sets (the note contains no explicit definition; with the other current meaning of semi-compactness, the proof would be wrong). If this is so, then the author's result follows at once (as pointed out in the note) from the same proposition restricted to compact spaces. Apparently, the author was not aware that the compact case is known [cf. Kuratowski, Topologie II, Warszawa-Wroclaw, 1950, p. 106; these Rev. 12, 517].

*M. Katětov* (Prague).

**Estill, Mary Ellen. Concerning a problem of Souslin's.** Duke Math. J. 19, 629–639 (1952).

A space is said to have property X if and only if it is not separable and does not contain uncountably many mutually exclusive domains. A Moore space is a space satisfying Axiom 0 and the first three parts of Axiom 1 of R. L. Moore [Foundations of point set theory, Amer. Math. Soc. Colloq. Publ., v. 13, New York, 1932]. F. B. Jones [Bull. Amer. Math. Soc. 45, 623–628 (1939), p. 628; these Rev. 1, 45] has shown that no linear space with property X is a Moore space. The author [Duke Math. J. 18, 623–629 (1951); these Rev. 13, 148] has shown that there exists a locally connected Moore space with property X and that there does not exist a locally connected Moore space with property X such that each two points can be separated by a finite point set.

In this paper the following two theorems are proved. 1. If there exists a locally connected space with property X such that each two points can be separated by a separable point set, then there exists a linear space with property X. 2. If there exists a linear space with property X, then there exists a locally connected Moore space with property X and such that each two points can be separated by a countable point set. The proof of the second theorem is difficult and accounts for most of the length of the paper. The paper was suggested by a problem proposed by Souslin [Fund. Math. 1, 223 (1920)].

*W. R. Utz* (Columbia, Mo.).

**Moise, Edwin E. Remarks on the Claytor imbedding theorem.** Duke Math. J. 19, 199–202 (1952).

The author gives a straightforward proof of the theorem that a cyclic Peano space is imbeddable in a 2-sphere provided that it does not contain a set homeomorphic with either of the "primitive skew graphs" $P_1$ and $P_2$ defined as follows: $P_1$ consists of the points $p_1, p_2, \cdots, p_5$ and the arcs $(p_ip_j)$, where $1\leqq i<j\leqq 5$, and where different arcs $(p_ip_j)$ intersect only at their end-points; $P_2$ consists of the points $p_1, p_2, p_3$ and $q_1, q_2, q_3$ and the arcs $(p_iq_j)$ where different

arcs $(p_i q_i)$ intersect only at their end-points. The proof is based on the recent theorem of R. H. Bing [Fund. Math. **36**, 303–318 (1949); these Rev. **12**, 348] to the effect that every Peano space has a complete sequence of brick partitionings.

*W. W. S. Claytor* (Washington, D. C.).

**Miščenko, E. F. On an elementary class of nonclosed sets and the duality theory for them.** Mat. Sbornik N.S. 31(73), 183–188 (1952). (Russian)

Aleksandrov has defined a duality domain [all references to Aleksandrov are to his paper, Mat. Sbornik N.S. **21**(63), 161–232 (1947); these Rev. **9**, 456] to be a collection $\Sigma$ of subsets of spheres $S^n$ of various dimensions such that: (1) if $A\subset S^n$ is in $\Sigma$, then $B=S^n-A\in\Sigma$; and (2) if $A\in\Sigma$ and $A'\subset S^m$ is homeomorphic to $A$, then $A'\in\Sigma$. A duality domain $\Sigma$ is defined by the author to be elementary if whenever $A\subset S^n$ is in $\Sigma$, then $\Delta_c{}^p(A,\mathfrak{A})$ and $\Delta_c{}^q(B,\mathfrak{B})$ are dual in the sense of Pontrjagin, where $p+q=n-1$, $\mathfrak{A}$ is an arbitrary discrete coefficient group, $\mathfrak{B}$ is its compact character group, and where $\Delta_c$ denotes the homology group of true cycles with compact carriers modulo those bounding on compact sets. It is the main purpose of the paper to prove that there exists an elementary duality domain which contains all stripped polyhedra, that is, all finite unions of pairwise disjoint open simplexes. The existence of such a domain for finite groups $\mathfrak{A}=\mathfrak{B}$ had been proved by Aleksandrov. A set $A\subset S^n$ is defined to be a two-sided homological retract if and only if $A$ possesses the properties $(J^p)$, $(a^p)$, and $(ur^p)$ of Aleksandrov for all $p\leqq n$ and all coefficient groups. The elementary duality domain is the set $\Sigma$ of all two-sided homological retracts. The theorem is obtained by proving that $A\in\Sigma$ implies $S^n-A\in\Sigma$, and then using results of Aleksandrov. *E. E. Floyd* (Charlottesville, Va.).

**Pontryagin, L. S. Classification of the mappings of an $(n+1)$-dimensional sphere into a polyhedron $K_n$ whose fundamental group and Betti groups of dimensions $2,\cdots,n-1$ are trivial.** Amer. Math. Soc. Translation no. 75, 54 pp. (1952).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 7–44 (1950); these Rev. **11**, 677.

**Sugawara, Masahiro. On families of continuous vector fields over spheres.** Math. J. Okayama Univ. **2**, 49–55 (1952).

It has been known for some time that for $n\equiv 3 \pmod 4$ the $n$-sphere, $S^n$, admits a set of three continuous vector fields independent at each point, i.e., a 3-field. G. W. Whitehead [Ann. of Math. (2) **47**, 779–785 (1946); these Rev. **8**, 167] has published a proof that for $n\equiv 3 \pmod 8$, $S^n$ does not admit a 4-field. Unfortunately, this proof made essential use of Pontrjagin's assertion, later proved erroneous, that $\pi_5(S^3)=0$. In this paper the author gives a new proof that the result of G. W. Whitehead is correct. Apparently he has not seen a paper of N. E. Steenrod and J. H. C. Whitehead [Proc. Nat. Acad. Sci. U. S. A. **37**, 58–63 (1951); these Rev. **12**, 847] in which a much more general result is established, viz., if $n=2^k(2r+1)-1$, then any set of $2^k$ continuous vector fields tangent to $S^n$ are somewhere dependent.

*W. S. Massey* (Providence, R. I.).

**Cockcroft, W. H. On the homomorphisms of sequences.** Proc. Cambridge Philos. Soc. **48**, 521–532 (1952).

L'étude du 3-type d'un complexe [cf. par ex. S. MacLane et J. H. C. Whitehead, Proc. Nat. Acad. Sci. U. S. A. **36**, 41–48 (1950); ces Rev. **11**, 450] conduit à étudier la situation algébrique suivante, qu'on rencontre aussi dans la théorie des $Q$-noyaux [Eilenberg et MacLane, Ann. of Math. (2) **48**, 326–341 (1947); ces Rev. **9**, 7]: soient deux groupes $A$ et $U$, un homomorphisme $j\colon A\rightarrow U$ et des opérations de $U$ (à gauche) dans $A$ telles que

$$j(u\cdot a)=u(ja)u^{-1}\quad (u\in U,\, a\in A),$$
$$j(a)\cdot a'=aa'a^{-1}\quad (a\in A,\, a'\in A).$$

On dit alors que $A$ est un $(U,j)$-module croisé [cf. J. H. C. Whitehead, Bull. Amer. Math. Soc. **55**, 453–496 (1949); ces Rev. **11**, 48]. Ceci détermine une suite exacte

$$0\rightarrow G\rightarrow A\xrightarrow{j} U\rightarrow Q\rightarrow 1,$$

où $G$ (noté additivement) est dans le centre de $A$, et $Q$ opère dans $G$. Cette suite définit, comme connu, une obstruction $\mathbf{k}\in H^3(Q,G)$.

L'auteur considère la notion (évidente) d'homomorphisme $\varphi\colon(A,U,j)\rightarrow(A',U',j')$ d'un $(U,j)$-module croisé dans un $(U',j')$-module croisé. Etant donné $(A,U,j)$ et $(A',U',j')$, l'auteur tente une classification de ces homomorphismes au moyen d'une notion d'"homotopie", qui est simple mais ne semble pas être celle qui permettrait de résoudre le problème de la classification des applications d'un 2-complexe dans un 2-complexe. (Ce dernier problème aurait été résolu par Whitehead et MacLane, d'après ce que dit l'auteur.) Dans la notion d'homotopie de H. Cockcroft le rôle essentiel est joué par une obstruction $\theta\in H^2(Q,G')$ attachée à tout couple d'homomorphismes $\varphi$, $\bar\varphi$ qui induisent les mêmes applications $G\rightarrow G'$ et $Q\rightarrow Q'$. *H. Cartan* (Paris).

**Rozenknop, I. Z. Homology groups of homogeneous spaces with a commutative stationary subgroup.** Doklady Akad. Nauk SSSR (N.S.) **85**, 1219–1221 (1952). (Russian)

Let $G$ be a compact, connected, semisimple Lie group and let $K$ be a closed abelian subgroup. Let $\mathfrak{a}$ be the Lie algebra of $G$ and $\mathfrak{h}$ the commutative subalgebra corresponding to $K$. This paper discusses the cohomology ring $H(\mathfrak{a},\mathfrak{h})$ over the rational field. If $\mathfrak{h}$ has (algebraic) dimension $m$, then, by picking out an arbitrary basis for $\mathfrak{h}$, a homomorphism $\phi$ may be set up between $R$, the ring of polynomials in $m$ variables with rational coefficients, and $H(\mathfrak{a},\mathfrak{h})$. Let $\phi(R)=\bar R$, and let the kernel of $\phi$ be $I$. Let $Y$ be the image of $H(\mathfrak{a},\mathfrak{h})$ in $H(\mathfrak{a})$ under the natural homomorphism and $H_0$ the kernel of this homomorphism. [Reviewer's note: The author does not use this homomorphism, but his definitions would be simplified by its introduction.] Then $Y$ may be identified with a vector subspace of $H(\mathfrak{a},\mathfrak{h})$ complementary to $H_0$. Using exterior multiplication, we may define a homomorphism of the tensor product $\bar R\otimes Y$ into $H(\mathfrak{a},\mathfrak{h})$ which is univalent.

The ideal $I$ admits a system of (homogeneous) independent generators $F_1,\cdots,F_m$, say. These generators are said to be fully independent (in the weak sense) if $\sum r_iF_i=0$, $r_i\in R$, implies $r_i\in I$. Then the main theorem asserts that, if $I$ admits a system of fully independent generators, then $H(\mathfrak{a},\mathfrak{h})\approx\bar R\otimes Y$ and the number of independent generators of $I$ of degree $q$ is equal to the dimension of the factor space of the primitive cohomology classes of $H(\mathfrak{a})$ of dimension $2q-1$, modulo those lying in $Y$. From this theorem, it is possible to obtain the Poincaré polynomial of the homogeneous space $G/K$. A partial converse of this theorem requires a further assumption about the Leray-Koszul sequences associated with the fibering $G\rightarrow G/K$. Among other results, the author asserts that the theorem quoted above holds if $G$ does not possess a pair of linearly independent primitive elements of the same dimension. The paper does not contain proofs. *P. J. Hilton* (Cambridge, England).

**Leray, Jean. Errata.** J. Math. Pures Appl. (9) **31**, 377–379 (1952).

This is a series of errata for the following papers: J. Math. Pures Appl. (9) **24**, 95–167, 169–199 (1945); 201–248 (1946); **29**, 1–139 (1950); these Rev. **7**, 468; **12**, 272.

***Eilenberg, Samuel, and Steenrod, Norman. Foundations of algebraic topology.** Princeton University Press, Princeton, New Jersey, 1952. xv+328 pp. $7.50.

Ceci est le premier de deux volumes consacrés à la théorie de l'homologie (il n'est question des groupes d'homotopie qu'incidemment, dans les Notes du Chap. I). On sait que la théorie axiomatique de l'homologie due à Eilenberg et Steenrod a déjà utilisée avec succès par divers auteurs; le présent livre en donne la première exposition complète. Cette conception axiomatique commande tout le développement de l'ouvrage.

Dans un premier bloc de 3 chapitres, les axiomes généraux sont exposés, et il est prouvé que le groupe de coefficients de la théorie détermine entièrement celle-ci sur la catégorie des espaces admettant une décomposition simpliciale finie. Aucun théorème d'existence n'est encore démontré: on n'exhibera explicitement des théories satisfaisant aux axiomes que dans les Chapitres VI, VII, et IX. Notons que, dès la fin du Chap. I, les groupes d'homologie des sphères sont déterminés et le théorème de Mayer-Vietoris établi comme simples conséquences des axiomes. Le Chap. II est consacré à l'étude géométrique des complexes simpliciaux. Au Chap. III, les axiomes conduisent, dans le cas d'un espace pourvu d'une décomposition simpliciale, à introduire les notions de chaîne, d'orientation, de bord, de cycle, etc.

Essentiellement, trois théories de l'homologie seront présentées dans la suite du livre: l'homologie des complexes simpliciaux, finis ou infinis (Chap. VI); la théorie singulière d'Eilenberg (Chap. VII); la théorie de Čech (Chap. IX et X). Auparavant, on a besoin de deux chapitres préparatoires (IV et V): le premier est consacré à un exposé sommaire des notions générales sur les catégories et les foncteurs (au sens d'Eilenberg-MacLane); le deuxième est un chapitre d'algèbre où l'on traite des "complexes de chaînes" en général, et où le lecteur trouvera aussi des notions sur les groupes abéliens libres, le produit tensoriel de deux modules $C$ et $G$, le module des homomorphismes Hom$(C, G)$, la comparaison des groupes d'homologie et de cohomologie relatifs à divers groupes de coefficients.

Les Chapitres VIII, IX, X constituent un bloc compact de 86 pages sur la théorie de Čech. Le Chap. VIII traite des limites inductives et limites projectives (direct limits, inverse limits), dont l'intérêt n'est d'ailleurs pas limité à la théorie de Čech. Au Chap. IX nous trouvons la théorie de Čech pour des espaces généraux, au moyen de recouvrements quelconques (Dowker). Le rapporteur pense qu'on aurait pu se contenter d'exposer la théorie de la cohomologie de Čech: en effet la théorie de l'homologie à coefficients dans un groupe compact $G$ (la seule qui satisfasse à l'"axiome d'exactitude") ne peut donner d'autres invariants topologiques que la cohomologie à coefficients dans le groupe discret, dual de $G$. Au Chap. X, on montre que, pour la catégorie des espaces compacts, l'"axiome de continuité" caractérise la théorie de Čech parmi toutes les théories possibles; puis on donne deux autres théories "de Čech", dont l'une est essentiellement la "cohomologie de Čech à supports compacts" pour un espace localement compact.

Le livre présente quelques traits particuliers dignes d'être signalés à l'attention des lecteurs. Il n'y a pas de liste bibliographique; mais à la fin de chaque chapitre on trouve des "Notes" qui contiennent éventuellement des indications sur le développement historique, et des compléments souvent très suggestifs. En outre, chaque chapitre est suivi d'exercices dont le but est double: ils permettent au lecteur de s'assurer qu'il a assimilé le texte, mais ils donnent aussi aux auteurs l'occasion de compléter celui-ci sur des points intéressants. Le livre est riche en diagrammes: la technique des diagrammes est devenue un auxiliaire presque indispensable en algèbre homologique. Signalons, entre autres détails intéressants: la théorie des produits dans les complexes simpliciaux (Chap. II), celle du "mapping cylinder" abstrait (Chap. V, démonstration de 13.3), resp. topologique (Chap. VII, exercice $C$), une démonstation modernisée des l'invariance topologique des groupes d'homologie simpliciaux d'un espace triangulé (Notes du Chap. VI); et enfin le Chapitre XI. Consacré aux applications topologiques, il contient notamment: 1) une jolie suite bien enchaînée de théorèmes (allant jusqu'au théorème de Brouwer sur l'invariance du domaine) qui découlent simplement du fait que l'application identique de la sphère $S^n$ est essentielle; 2) des conséquences topologiques de la notion de degré d'une application de $S^n$ dans $S^n$, avec application au théorème fondamental de l'algèbre et à ses généralisations.

Le volume 2 doit contenir des développements sur le calcul pratique de l'homologie d'un espace muni d'une décomposition cellulaire, six chapitres sur la théorie des produits (cross, cup et cap product), et une axiomatisation de l'homologie à coefficients locaux. Aucune exposition de la théorie des puissances de Steenrod ne semble devoir figurer dans ce traité. *H. Cartan* (Paris).

## GEOMETRY

**Thébault, Victor. The area of a triangle as a function of its sides.** Scripta Math. **18**, 151–161 (1952).

**Bilo, J. Sur une relation affine remarquable entre deux triangles.** Mathesis **61**, 161–168 (1952).

**Toscano, L. Relations métriques de la géométrie du triangle par rapport aux centres isogones et isodynamiques.** Mathesis **61**, 174–178 (1952).

**Goormaghtigh, R. Sur le pôle multi-angulaire d'un système de droites par rapport à un triangle.** Mathesis **61**, 200–204 (1952).

**Blanchard, René. Triangles inscrits et circonscrits à deux hyperboles équilateres concentriques.** Mathesis **61**, 193–199 (1952).

**Hajós, G. Über die Feuerbachschen Kugeln merhdimensionaler orthozentrischer Simplexe.** Acta Math. Acad. Sci. Hungar. **2**, 191–196 (1951). (Russian summary)

Using barycentric (areal, voluminal, $\cdots$) coordinates Egervary has shown that the nine-point circle of a triangle has its analogs in connection with orthocentric simplexes [same Acta **1**, 5–16 (1950); these Rev. **12**, 629]. In the present paper those results, and some additions, are obtained by means of vectors. *N. A. Court* (Norman, Okla.).

**Dougall, John. The double six of lines and a theorem in Euclidean plane geometry.** Proc. Glasgow Math. Assoc. 1, 1–7 (1952).

Representing the lines of projective 3-space by the points of a quadric fourfold $Q$ in projective 5-space, the author shows that Schläfli's "double six" theorem is equivalent to the following: If a simplex is inscribed in $Q$ in such a way that five of the six bounding hyperplanes touch $Q$, then the sixth hyperplane likewise touches $Q$. By regarding $Q$ as a 4-sphere in complex Euclidean 5-space, and making some projections, he relates this to a simple theorem of plane geometry: If a triangle is inscribed in a circle and circumscribed about an ellipse, then the squared product of the radius of the circle and the minor axis of the ellipse is equal to the product of the powers of the two foci with respect to the circle. He deduces this from the classical invariant relation $\Theta^2=4\Delta\Theta'$, thus providing a new proof for the double six theorem itself. *H. S. M. Coxeter* (Toronto, Ont.).

***Breidenbach, Walter. Das Delische Problem. (Die Verdoppelung des Würfels.)** 2te Aufl. B. G. Teubner Verlagsgesellschaft, Leipzig, 1952. 59 pp. DM 2.30.

**Giovanardi, Mario. Sulla prospettiva di una superficie di rotazione e della sua separatrice di ombra.** Rend. Accad. Sci. Fis. Mat. Napoli (4) 14 (1946–47), 205–211 (1948).

**Giovanardi, Mario. Sulla restituzione prospettiva col quadro inclinato.** Rend. Accad. Sci. Fis. Mat. Napoli (4) 15 (1948), 105–114 (1949).

**Giovanardi, Mario. Il metodo della tripla proiezione ortogonale e sua applicazione alla prospettiva lineare cilindrica.** Rend. Accad. Sci. Fis. Mat. Napoli (4) 16 (1949), 85–125 (1950).

**Giovanardi, Mario. Il metodo della proiezione isometrica.** Rend. Accad. Sci. Fis. Mat. Napoli (4) 17 (1950), 73–102 (1951).

**Lorent, H. Courbes construites à partir de deux coniques.** Bull. Soc. Roy. Sci. Liège 21, 232–246 (1952).

**Wunderlich, Walter. Euklidische und nichteuklidische *D*-Linien auf Quadriken.** Ann. Mat. Pura Appl. (4) 33, 145–164 (1952).

"Die auf einer Fläche 2. Ordnung verlaufenden Kurven, deren Schmiegkugeln die Fläche berühren, werden hier erstmalig vom synthetischen Standpunkt aus betrachtet. In einfachster Weise ergeben sich dabei nicht nur die wenigen bekannten, sondern auch zahlreiche neue geometrische Eigenschaften dieser im allgemeinen transzendenten Kurven, die gleichzeitig einer darstellend-geometrischen Behandlung zugänglich gemacht werden. Die unschwer durchzuführende Übertragung der im euklidischen Raum herrschenden Beziehungen auf einen Raum mit Cayley-Kleinscher Metrik weist auf manche interessanten Zusammenhänge hin." (Author's abstract.) *P. Scherk.*

**Fiore, Anna. Elementi della teoria delle coniche e delle polarità piane biduali nella 1ª rappresentazione complessa.** Rend. Accad. Sci. Fis. Mat. Napoli (4) 15 (1948), 151–153 (1949).

**Ströher, Wolfgang. Die Ableitung der trigonometrischen Formeln im Poincaréschen Modell der hyperbolischen Geometrie.** Elemente der Math. 7, 130–135 (1952).

**Leisenring, Kenneth. The natural map of the hyperbolic plane into the Euclidean circle.** Michigan Math. J. 1, 5–10 (1952).

The earliest proof that the consistency of Euclidean geometry implies the consistency of hyperbolic geometry was supplied by Beltrami in 1868, when he represented the lines of the hyperbolic plane by chords of a Euclidean circle. In the present paper this representation is derived from the well-known fact that, in hyperbolic 3-space, the lines and planes of a bundle of parallels (all passing through a fixed point at infinity, $C$) behave like the points and lines of a Euclidean plane. Those lines and planes of the bundle which meet a plane $\delta$ (of general position) are taken to represent the points and lines in which they meet $\delta$. Thus the hyperbolic plane $\delta$ is mapped on part of the Euclidean plane, namely the interior of a circle. The points on this circle correspond to those lines of the bundle which, being parallel to $\delta$, generate a cone of revolution (or "cylinder") with vertex $C$ and axis perpendicular to $\delta$. Parallel lines in $\delta$, being sections of planes meeting in a generator of the cone, are represented in the Euclidean circle by chords meeting on the circumference.

Regarding the hyperbolic space as the interior of a quadric in real projective space, we see that, since $\delta$ and the cone both meet the absolute quadric in the same conic, perpendicular lines in $\delta$ are sections of conjugate planes with respect to the cone, and are therefore represented by conjugate lines with respect to the Euclidean circle. The author obtains these descriptive results, and many consequent metrical results, by methods that would have been available to Lobačevskiĭ. *H. S. M. Coxeter* (Toronto, Ont.).

***Mullins, Edgar Raymond, Jr. A straight line plane with preassigned circles.** Abstract of a thesis, University of Illinois, Urbana, Ill., 1952. i+7+i pp.

**Pickert, Günter. Der Satz vom vollständigen Viereck bei kollinearen Diagonalpunkten.** Math. Z. 56, 131–133 (1952).

If $ABCD$ is a quadrilateral, the theorem of the complete quadrilateral says that $P=AB\cap CD$, $Q=AC\cap BD$, and $S=PQ\cap BC$ determine $T=PQ\cap AD$ uniquely. If the third diagonal point $R=AD\cap BC$ lies on $PQ$, then $S=T=R$. It is shown here, assuming the theorem of the complete quadrilateral, that if the diagonal points of one quadrilateral are collinear, then the diagonal points of every quadrilateral are collinear. *Marshall Hall* (Columbus, Ohio).

**Skopec, Z. A. Curves defined by configurations of Desargues.** Doklady Akad. Nauk SSSR (N.S.) 85, 277–280 (1952). (Russian)

It is shown that in general there will be a unicursal curve of sixth degree through the ten points of a Desargues configuration. But if the Desargues configuration is specialized by further incidences (e. g., if the center of perspectivity lies on the axis of perspectivity), then the unicursal curve will be of lower degree. *Marshall Hall* (Columbus, Ohio).

**Lagrange, René. Métrique anallagmatique.** Acta Math. 86, 259–295 (1951).

Let $H_{n-p}$ denote an $(n-p)$-dimensional hypersphere in conformal $n$-space $[p=1, 2, \cdots, n]$; $H_{n-1}$=sphere, $H_1$=circle, $H_0$=bi-point. Let $U, V, \cdots$ be spheres. If $U$ has the center $A$ and the radius $R$, then the point-function $U=U(P)=(\overline{AP}^2-R^2)/2R$ is associated with $U$ ($R^2$, $A$ and $P$ real). Define the conformal distances $[UV]=i\sin\sphericalangle(U, V)$

and $[PU]=U(P)/\operatorname{sgn} R$ where $\operatorname{sgn} R=\operatorname{sgn} R/i$ if $R$ is imaginary. Let $\alpha$ be an $H_{n-p}$. The intersection $\alpha'$ of the spheres orthogonal to $\alpha$ is the focal $H_{p-2}$ of $\alpha$. Focality is symmetric. Let $U_1, \cdots, U_p$ be an orthogonal basis of the pencil $\pi$ of the spheres through $\alpha$; i.e., the $U_k$'s are mutually orthogonal and the spheres of $\pi$ are identical with the linear combinations $U=\sum\lambda_k U_k$ where $\sum\lambda_k^2=1$. The distance $[P\alpha]$ is defined to be the stationary value of $[PU]$ as $U$ ranges through $\pi$. Then

$$[P\alpha]^2=-[P\alpha']^2=\sum[PU_k]^2=[PU_0]^2$$

where $U_0\subset\pi$ intersects the $H_{n-p+1}$ through $P$ and $\alpha$ orthogonally along $\alpha$. If $\alpha$ is real, then $[P\alpha]=\max_{U\subset\pi}[PU]$.

Let $\beta$ be an $H_{n-q}[p\leqq q\leqq n]$. Let $\beta'$ be the focal $H_{q-2}$ of $\beta$ and let $V_1, \cdots, V_q$ be an orthogonal basis of the pencil $\nu$ of spheres through $\beta$. Suppose there is an $H_{n-r}$ but not an $H_{n-r-1}$ which contains both $\alpha$ and $\beta$. The author defines the conformal distances $[\alpha\beta]$ of $\alpha$ and $\beta$ to be the $p-r$ stationary values of $[UV]$ as $U$ and $V$ range through $\pi$, respectively $\nu$. They are related to the eigen-values of a certain matrix. We have $[\alpha\alpha']=i$; $[\alpha\beta]=[\alpha'\beta']$. Apart from certain distances 0 and $-1$, the distances $[\alpha\beta]$ and $[\alpha\beta']$ can be grouped in pairs such that $[\alpha\beta]^2+[\alpha\beta']^2=-1$. The $[\alpha\beta]$'s are the stationary values of the (unique) conformal distance of $\alpha$ to a bi-point on the intersection of $\beta$ with a variable $H_{n-p+1}$ through $\alpha$ (cf. below). They are also (essentially) the stationary values of the distances between two co-cyclic bi-points, one moving on $\alpha$ the other one on $\beta$. The circles through the pairs of bi-points which belong to these values are the common normal circles of $\alpha$ and $\beta$. Special cases:

$$[U\beta]^2=q-1+\sum_1^q[UV_k]^2.$$

If $\beta$ is a bi-point and $r=p-1$, then $[\alpha\beta]^2=\sum_1^p[U_k\beta]^2$. If $\alpha=(A, A')$ and $\beta=(B, B')$ are co-cyclic bi-points, then

$$[\alpha\beta]^2=4\frac{\overline{AB}\cdot\overline{AB'}\cdot\overline{A'B}\cdot\overline{A'B'}}{\overline{AA'}^2\cdot\overline{BB'}^2}$$

where bars indicate euclidean distances. *P. Scherk.*

**Chamard, Lucien. Sur la distance d'un point variable à un ensemble fixe.** Gaz. Mat., Lisboa **13**, no. 51, 1–3 (1952).

Définitions et notations. $E$: ensemble fermé de l'espace euclidien à trois dimensions. $\rho=\rho(M)$: distance d'un point $M$ variable à $E$. $\varpi(M)$: ensemble des points $P$ de $E$ (projections) à distance $\rho$ de $M$. Point de multifurcation: point $M$ admettant au moins deux projections sur $E$. $\phi(M)$: ensemble des rayons $MP$ (projetantes). $Mx$: demi-droite issue de $M$ extérieure à $\phi(M)$. $\varphi$: distance angulaire de $Mx$ à $\phi(M)$. $\Gamma^{\varphi}{}_{Mx}$: cône d'appui de $\phi(M)$ d'axe $Mx$ (donc de demi-angle au sommet $\varphi$).

Théorèmes. (1) Si $M_1, M_2, \cdots, M_n, \cdots$ est une suite de points convergeant vers $M$ suivant $Mx$, $M_1x_1, \cdots, M_nx_n, \cdots$ une suite de rayons (demi-droites) telle que $M_nx_n \in \phi(M_n)$ pour $n=1, 2, \cdots$, et convergeant vers un rayon $MA$, alors $MA \in \Gamma^{\varphi}{}_{Mx}\cdot\phi(M)$. Inversement si $MA$ est un rayon quelconque de $\Gamma^{\varphi}{}_{Mx}\cdot\phi(M)$, il existe une suite $M_1x_1, \cdots, M_nx_n, \cdots$ du type précédent convergeant vers $MA$. (2) La dérivée de $\rho(M)$ dans la direction $Mx$, dont l'existence a été établie par R. de Misès [C. R. Acad. Sci. Paris **205**, 1353–1355 (1937)] est $=-\cos\varphi$. (3) Si $M$ est un point de multifurcation à partir duquel $\rho(M)$ puisse croître, $M'$ le point sur la sphère de centre $M$ et de rayon $\delta>0$ à distance maximale de $E$, pour $\delta$ suffisamment petit, $M'$ est un point de multifurcation.

Les démonstrations font appel à l'acquiescement visuel du lecteur. La version du Théorème 3 ici donnée correspond à ce que le rapporteur croit avoir compris à l'aide de la "démonstration".

L'étude de la fonction $\rho(M)$ dans le cas d'un ensemble plan se trouve dans la note du rapporteur [Rev. Sci. (Rev. Rose Illus.) **77**, 493–496 (1939); ces Rev. **1**, 109]. Les méthodes employées s'appliquent au cas d'un espace euclidien de dimension quelconque. *C. Y. Pauc.*

## *Convex Domains, Extremal Problems, Integral Geometry*

***Pogorelov, A. V. Izgibanie vypuklyh poverhnosteĭ.** [Deformation of convex surfaces.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 184 pp. 6.75 rubles.

The following terminology will be used: A closed (open) convex surface is the boundary of a bounded (unbounded) convex set with interior points in $E^3$. A convex surface is a connected open subset of a closed (open) convex surface. Two convex surfaces are isometric if they are intrinsically isometric, congruent if one can be carried into the other by a motion of $E^3$, reflection admitted. A convex surface $C$ is rigid if every convex surface isometric to $C$ is congruent to $C$.

The investigations of the present book mostly belong to classical differential geometry but go beyond it in one very essential point, which has not been appreciated and will therefore be explained by an example: The fact that an ellipsoid $E$ is rigid means in standard differential geometry that any surface of class $C^2$ with positive Gauss curvature isometric to $E$ is congruent to $E$, leaving the possibility that there are convex surfaces not of class $C^2$ isometric but not congruent to $E$. Now A. D. Alexandroff has proved rather simply that on any surface of bounded curvature (i.e., on all Borel sets the ratio of the integral curvature, which can be defined for all convex surfaces, to the area is uniformly bounded) a point where the surface is not differentiable is an interior point of a straight edge; hence a closed convex surface with bounded curvature is differentiable everywhere. A convex surface isometric to $E$ is therefore at least of class $C^1$ and never pathological. The existence of such a surface not congruent to $E$ could therefore materially detract from the geometric value of the standard theorem, or at least alter its character entirely. Thus there is no way of getting out of the fact that the behavior of surfaces of class less than 2 is relevant for standard differential geometry.

The author has recently outlined a proof for the fact that any two isometric closed convex surfaces are congruent [Doklady Akad. Nauk SSSR (N.S.) **79**, 739–742 (1951); these Rev. **13**, 271]. This solves, of course, all rigidity questions for closed surfaces and outdates parts of the present book, which was evidently printed before this result was found, because there are only two brief references to it at places where a posteriori insertion was trivial.

Nevertheless the book is full of interesting results, the principal of which are as follows: If a convex surface has an intrinsic metric which is $k$ times differentiable, $k\geqq 5$, or analytic (i.e., $E(u, v)$, $F(u, v)$, $G(u, v)$ have all derivatives of order less than or equal to $k$ or are analytic) and has positive Gauss curvature, then the surface itself is at least $k-1$ times differentiable or analytic, i.e., the components

$x(u, v)$, $y(u, v)$, $z(u, v)$ of a vector from a fixed point to the point $(u, v)$ on the surface are $k-1$ times differentiable or analytic. This implies, for instance, the rigidity of the ellipsoid within the class of all closed convex surfaces and in conjunction with A. D. Alexandroff's theorem on the existence of a convex surface with an abstractly given metric it furnishes a new proof for the theorem of H. Weyl and H. Lewy on the analyticity of a convex surface with a given analytic line element of positive curvature on the sphere. Further results are: If $F$ is a $k$ times differentiable, $k>5$, or analytic surface of positive Gauss curvature, then any surface isometric to $F$ is (at least) $k-2$ times differentiable or analytic. A given $k$ times differentiable metric, $k\geqq 5$, of positive Gauss curvature can be locally realized as a $k-1$ times differentiable surface. There follow theorems on the rigidity of open convex surfaces and an interesting theorem on non-rigidity: If $F_1$ and $F_2$ are two equally oriented isometric open surfaces which are $k$-times differentiable, $k\geqq 5$, and have positive Gauss curvature and whose spherical images are bounded by smooth curves (not degenerating to a point), then $F_1$ can be deformed into $F_2$, i.e., a continuous family of isometric surfaces containing both $F_1$ and $F_2$ exists.

*H. Busemann* (Los Angeles, Calif.).

**Pogorelov, A. V. Quasi-geodesic lines on a convex surface.** Amer. Math. Soc. Translation no. 74, 45 pp. (1952).

Translated from Mat. Sbornik N.S. **25**(67), 275–306 (1949); these Rev. **11**, 201, 871.

**Molnár, J. Inhaltsabschätzung eines sphärischen Polygons.** Acta Math. Acad. Sci. Hungar. **3**, 67–70 (1952). (Russian summary)

Habicht and van der Waerden [Math. Ann. **123**, 223–234 (1951), pp. 228–231; these Rev. **13**, 154] gave a rather complicated proof that $T_n\geqq(n-2)T_3$, where $T_n$ denotes the area of a spherical $n$-gon of side $a$ such that the spherical distance between any two vertices is at least $a$. The author proves this more simply, by showing that the $n$-gon can be dissected into at least $n-2$ spherical triangles, each having area at least $T_3$. *H. S. M. Coxeter* (Toronto, Ont.).

**van der Waerden, B. L. Punkte auf der Kugel. Drei Zusätze.** Math. Ann. **125**, 213–222 (1952).

The author continues his investigation of the diameter $d$ of the smallest sphere on which $N$ nonoverlapping circles of (straight) diameter 1 can be drawn. Improving the inequality (17) of Habicht and van der Waerden [Math. Ann. **123**, 223–234 (1951), p. 234; these Rev. **13**, 154], he finds that

$$\frac{\sqrt{3}}{2\pi}N\leqq d^2\leqq\frac{\sqrt{3}}{2\pi}N+3\left(\frac{N}{4\pi}\right)^{2/3}+3\left(\frac{N}{4\pi}\right)^{1/3}.$$

He simplifies the proof of Schütte and van der Waerden [ibid. **123**, 96–124 (1951), pp. 102–104; these Rev. **13**, 61] that, when $N=7$, $d^2=2.532\cdots$. By modifying an arrangement of points due to Rutishauser [Comment. Math. Helv. **17**, 327–331 (1945); these Rev. **7**, 164], he shows that, when $N=20$, $d^2\leqq 6.18\cdots$. (Rutishauser's result was $d^2\leqq 6.37\cdots$.) He then proves that, when $N=42$, $d^2\leqq 13.095\cdots$, an upper bound differing by only 11% from Fejes Tóth's lower bound

$$(1-\cos 63°)/(\tfrac{1}{2}-\cos 63°)=11.868\cdots.$$

Finally, he shows that, when $N=122$, $d^2\leqq 35.41$, differing by less than 5% from

$$(1-\cos 61°)/(\tfrac{1}{2}-\cos 61°)=33.91\cdots.$$

*H. S. M. Coxeter* (Toronto, Ont.).

**Eggleston, H. G., and Taylor, S. J. On the size of equilateral triangles which may be inscribed in curves of constant width.** J. London Math. Soc. **27**, 438–448 (1952).

It is shown that among the greatest equilateral triangles which may be inscribed in different curves of constant width 1 that one inscribed in a "Reuleaux pentagon" is the least.

*L. Fejes Tóth* (Veszprém).

## *Algebraic Geometry*

***Godeaux, Lucien. Introduction à la géométrie supérieure.** 2ème éd. Masson & Cie, Paris, 1946. 149 pp.

This textbook is designed for an intermediate course in algebraic geometry to follow elementary projective geometry. As stated in the preface, the topics covered are intended to be studied by all mathematical students in the University of Liège and to serve as a basic course for those specializing in geometry. The fundamental concepts of projective space of one, two and three dimensions are treated.

The first chapter contains a detailed discussion of series of groups of points, correspondences, and polar groups on a line. Then follows a chapter on the properties of plane curves of any order and one on cubics and quartics. The study of cubics includes elliptic cubics, singularities, invariants and methods of generation. Non-singular quartics only are treated.

Almost half of the book is devoted to three-space. In chapter four the author deals with the properties of algebraic surfaces of any order, including singularities, polar surfaces and ruled surfaces. This chapter also contains a brief treatment of space curves. In chapter five, curves on a quadric are studied. The quadric $Q$ and its curves are projected from a point of $Q$ on a plane. Curves of order $n$ on $Q$ are classified for $n\leqq 6$. In the final chapter, cubic surfaces are treated in considerable detail, including representation on a plane, singularities, curves on cubics, generation of cubics, and ruled cubics.

The book is well written. The arrangement and printing leave nothing to be desired. It is an excellent introduction to algebraic geometry. *T. R. Hollcroft* (Aurora, N. Y.).

**Gröbner, W. Applicazione del calcolo vettoriale alla geometria algebrica.** Ann. Univ. Ferrara. Parte I. **8** (1948–50), 63–68 (1951).

A new method is exposed of applying vectorial calculation to algebraic geometry. If $\mathbf{v}(a_1, \cdots, a_n)$ and $\mathbf{w}(b_1, \cdots, b_n)$ are vectors, the product $\mathbf{vw}$ is defined as the set of $\frac{1}{2}n(n+1)$ numbers $(a_1b_1, a_1b_2+a_2b_1, \cdots, a_nb_n)$, called a vector of degree two. As an example it is shown that an algebraic curve in $S_3$ containing a point system $g_2{}^1$ can be given by the single vector equation $\mathbf{v}^2+\mathbf{a}_1(\lambda)\mathbf{v}+\mathbf{a}_2(\lambda)=0$, where $\mathbf{a}_1$ is an ordinary vector and $\mathbf{a}_2$ a vector of degree two.

*J. Haantjes* (Leiden).

**Purcell, Edwin J. Noninvolutorial Cremona transformations in [$n$].** Proc. Amer. Math. Soc. **3**, 335–347 (1952).

Verf. bestimmt eine ausgedehnte Klasse von nicht-involutorischen Cremona-Transformationen zwischen zwei $n$-dimensionalen projektiven Räumen. Er geht aus von einer geordneten Partition $\sum_{i=1}^{m} r_i=n$ von $n$. Es sei $S_0=0$, $S_i=S_{i-1}+r_i$. Er betrachtet eine Folge von Gleichungs-

systemen

$$\sum_{j=S_{l-1}+1}^{S_l+1} a_{ij}\bar{x}_j=0 \quad (i=S_{l-1}+1, \cdots, S_l, l=1, 2, \cdots, m).$$

Darin seien die $a_{ij}$ Linearformen in der anderen Variabelnreihe $x_1, \cdots, x_{n+1}$ und homogene Formen in den Variablen $\bar{x}_i$ mit $i \leqq S_{l-1}+1$. Setzt man noch voraus, dass die in Frage kommenden Unterdeterminanten nicht verschwinden, so ist dies System nach jeder der beiden Variabelnreihen auflösbar und stellt eine Cremona-Transformation dar. Verf. bestimmt ihr Fundamentalsystem. Er fasst alle Transformationen zu einer Klasse zusammen, die zu einer Partition gehören. Für $n=2$ erhält er mit $2=1+1$ die quadratischen, für $2=2+0$ die de Jonquières-Transformationen. Für $n=3$ erhält er die meisten bekannten Transformationen mit der Partition $3=3+0$. Er untersucht aber dann auch die Transformationen, die zu den übrigen Partitionen von 3 gehören. *O.-H. Keller* (Halle).

**Longo, C. Sulle trasformazioni puntuali fra due $S_r$ nell'intorno di una coppia a Jacobiano nullo di caratteristica massima.** Ann. Scuola Norm. Super. Pisa (3) **5**, 161–173 (1951).

Gegeben sei eine Beziehung $T$ zwischen den Punkten zweier Räume $S_r$ von der nur die Differenzierbarkeit im allgemeinen Punkt vorausgesetzt sei. Es seien $O$ und $O'$ entsprechende Punkte, und in $O$ verschwinde die Funktionaldeterminante, ohne dass alle ihre Unterdeterminanten verschwinden. Verf. fragt nach Bedingungen dafür, dass $T$ durch eine Cremona-Transformation approximierbar sei, die bis zur Umgebung $n$-ter Ordnung von $O$ mit $T$ übereinstimmt. Er findet: Ist $O$ ein einfacher Punkt der Jacobischen Fläche $J$, so ist die notwendige und hinreichende Bedingung, dass es eine Kalotte mit dem Mittelpunkt $O$ der Dimension $r-1$ und der Ordnung $n-1$ in $J$ gibt, der eine Kalotte $(r-2)$-ter Dimension mit dem Mittelpunkt $O'$ entspricht. Verf. kann sein Ergebnis auch auf den Fall übertragen, dass $O$ für $J$ $s$-fach ist. Für $r=2$, 3 und $n=2$ ist der Satz bereits durch Villa und Vaona [Boll. Un. Mat. Ital. (3) **5**, 101–107 (1950); diese Rev. **12**, 438] bekannt. *O.-H. Keller.*

**Yeh, Mo. Foci of plane curves.** Univ. Washington Publ. Math. **3**, 109–116 (1952).

A focus of a curve is the point of intersection of two isotropic tangents and a corresponding directrix is the chord of contact of two isotropic tangents. This paper exhibits families of algebraic curves whose properties concerning foci and directrices generalize the corresponding facts for conics. It is shown that under appropriate conditions, the image of a focus of a curve under a conformal transformation is a focus of the image of the curve. *G. B. Huff.*

**Tibiletti, Cesarina. Piani tripli e piani quadrupli con la stessa curva di diramazione.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 537–543 (1952).

Lorsque l'on construit la trace caractéristique d'une courbe, le problème de la construction des plans multiples admettant cette courbe pour courbe de diramation, revient à la recherche des distributions d'échanges sur les diverses déterminations de la fonction représentée par le plan multiple, qui vérifient les conditions d'invariance d'Enriques. L'étude systématique des distributions possibles dans les cas énumérés ci-après, conduit l'A. aux conclusions suivantes: La $C_6$ avec 9 cuspides est de diramation pour un plan triple et pour 4 plans quadruples birationnellement distincts. La $C_6$ avec 6 cuspides situés sur une conique est diramante d'un seul plan triple. La $C_6$ avec 6 cuspides et 4 points doubles est diramante d'un plan triple et d'un quadruple. La $C_6$ avec 9 cuspides et 12 points doubles est diramante d'un seul plan quintuple. Si un plan triple a une courbe diramante d'ordre $2p+4$, il y a au plus $2^{2p}-1$ plans quadruples ayant même courbe de diramation; le plan triple est résolvant cubique de ces plans quadruples. *B. d'Orgeval* (Alger).

**Bílek, J. On a construction of an involution of de Jonquière of the fifth degree.** Časopis Pěst. Mat. **76**, 141–144 (1951). (Czech)

**Riccio, Maria Teresa. La varietà $W_3{}^6$ dell'$S_7$ riemanniana dell'$S_1$ di 1ª specie legato all'algebra di Study.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **14** (1946–47), 166–170 (1948).

**Riccio, Maria Teresa. Il sistema $\infty^6$ di $V_3{}^3$ di $S_6$ rappresentante le catene unidimensionali dell' $S_1$ di 1ª specie di Study.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **15** (1948), 88–91 (1949).

**Fadini, Angelo. La riemanniana del piano triduale.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 300–309 (1951).

**Tosques, Emilia. Caratterizzazione del sistema lineare di ipersuperficie cubiche dell'$S_8$ immagine complessa delle catene bidimensionali dell'$S_1$ triduale.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **16** (1949), 149–151 (1950).

**Godeaux, Lucien. Sopra alcune superficie algebriche dell'iperspazio.** Ann. Univ. Ferrara. Parte I. **8** (1948–50), 157–163 (1951).

Determination of the canonical system of the surface intersection of a cone projecting a generalized surface of Veronese and a general form. Cyclical involutions that may belong to such a surface. *Author's summary.*

**Vaccaro, Giuseppe. Questioni riguardanti i flessi delle ipersuperficie.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 322–334 (1950).

La simple considération de l'équation d'une hypersurface $V$ tout le long d'une courbe, lieu de points inflexionnels, montre que le long de tout lieu de tels points, l'hyperplan tangent est fixe; il en résulte, si la $V$ est algébrique et que ce lieu non linéaire ait sa dimension égale à celle de $V$ moins 1, l'ordre de $V \geqq 6$; si cet ordre $\leqq 5$, toute variété de points inflexionnels de dimension $r-1$ est linéaire. L'A. établit la forme du développement de l'équation de la $V_n{}^3$ au voisinage d'un point inflexionnel possédant un certain nombre de points analogues infiniment voisins, ou une calotte de tels points; s'il y a sur une $V_n{}^3$, $n-1$ points $\infty^1$ voisins alignés, leur droite est lieu de tels points et contient $n-1$ points bihyperplanaires; résultat similaire pour les calottes. Il en résulte la condition nécessaire et suffisante; pour que la $V_n{}^3$ possède une calotte linéaire à $r-1$ dimensions d'ordre $h$, lieu de points inflexionnels, à savoir que l'hyperplan tangent au centre possède un point triple dont le cône tangent soit formé d'un plan triple, contenant le voisinage d'ordre $h+2$ de sa section. Application aux cas $n=3$, 4. De même si la $V$ possède en 0 et en $k$ points $\infty^1$ voisins en directions distinctes des points inflexionnels, tout le voisinage du 1er ordre de 0 appartient à l'espace linéaire ainsi déterminé, qui

est lieu de tels points; propriété similaire pour les calottes. En particulier, sur une $V_n^2$ algébrique d'ordre $n$ s'il y a $k$ éléments rectilignes $E_{n-2}$ de points inflexionnels le $S^k$ déterminé appartient à $V_n^2$, est lieu de points inflexionnels, il existe une $V_{n-1}^{k-1}$ de points bihyperplanaires; de même si une $V_n^2$ algébrique possède $k(n-1)$ points bihyperplanaire situés $(n-1)$ à $(n-1)$ sur $k$ droites distinctes, leur $S^k$ appartient à $V_n^2$, y est lieu de points inflexionnels et contient une $V_{n-1}^{k-1}$ de points bihyperplanaires. *B. d'Orgeval* (Alger).

**Gaeta, Federico. Sulle rigate doppie di genere lineare assoluto $p^{(1)}=1$.** Rend. Accad. Naz. dei XL (4) 2, 23–63 (1951).

In his posthumous work "Le superficie algebriche" [Zanichelli, Bologna, 1949; these Rev. 11, 202]. Enriques reviewed in some detail the classification of regular surfaces of absolute linear genus $p^{(1)}=1$, and more particularly (p. 258) those, with $P_2>1$, which contain a linear pencil $|C|$ of elliptic curves. These latter surfaces are classified according to (i) their genus $p$ $(=p_a)$, (ii) their determinant $d$, which is the least number of points rationally determinable on a $C$ in terms of the parameter fixing $C$, and (iii) their numbers of curves of $C$ which degenerate in the various arithmetically possible ways into multiples of a single curve. The memoir now under review is for the most part a careful re-examination and revision of the theory of such surfaces for the cases $d=1$ and $d=2$; but it also contains applications to the theory of surfaces which admit an infinite discontinuous group of self-transformations.

As regards the case $d=1$, the author shows that, for given $p$, all surfaces of the type in question, defined to within birational transformation, generate a single irreducible family $\mathfrak{C}_1(p)$, depending on $10p+8$ moduli; also that the generic surface of $\mathfrak{C}_1(p)$ (freed from exceptional curves) can be represented without essential exception on a double rational normal cone $\Phi^{2p+2}[2p+3]$ with a branch curve consisting of the neighborhood of the vertex of $\Phi$ together with the section of $\Phi$ by a general cubic primal. This result differs from Enriques' classification of the same surfaces into two families, of modular freedom $8p+7$ and $8p+5$ respectively, representable by two types of double plane. By projecting his double cones into double planes, however, the author exhibits Enriques' two families as being visibly subordinate to his single family of modular freedom $10p+8$. The most difficult and delicate part of the whole investigation is that in which it has to be shown that the $(10p+8)$-fold family described above is not itself subordinate to any larger family of surfaces of the same general type.

For $d=2$, on the other hand, the results which the author here obtains by his own method now ageee with those of Enriques insofar as concerns the modular freedom of the families exhibited. If $F$ is a surface, free from exceptional curves, for which $d=2$, the pencil of elliptic curves $|C|$ on $F$ now has a bisecant curve or curves $B$; and it may also possess a number $s=P_2-2p+1$ of double curves $2C_i\equiv C$ $(i=1,\cdots,s)$. Considering first the case $s=0$, and defining $\pi$ to be the virtual genus of the bisecant curves $B$ of minimum grade $m$, so that $m=-2(p-\pi)=-2v$ say, where $v>0$ or $v=0$ according as $B$ is isolated or belongs to a pencil $|B|$, the author proves: (i) If $v>0$, the regular surfaces for which $p^{(1)}=1$, $d=2$, $s=0$, and $p$, $\pi$ are given form a single family $\mathfrak{C}_2(p,\pi)$ of modular freedom $9p+\pi+9$, and the generic surface of the family is representable on a double rational normal cone $\Phi^{\nu}[\nu+1]$, with a branch curve which meets the generators of $\Phi$ in four points and passes $2\pi+2$ times (in distinct directions) through the vertex and has no other singular points; (ii) if $v=0$, the regular surfaces for which $p^{(1)}=1$, $d=2$, $s=0$, and $p(=\pi)$ is given form a single family $\mathfrak{C}_2(p,p)$ of modular freedom $10p+8$, and the generic surface of the family is representable without exception on a double proper quadric, with a $(4,2p+2)$ curve on this quadric as branch curve. For $2\leqq\pi\leqq p$, it is shown that $\mathfrak{C}_2(p,\pi-2)$ is always a subordinate family of $\mathfrak{C}_2(p,\pi)$, and this gives two distinct leading families $\mathfrak{C}_2(p,p)$ and $\mathfrak{C}_2(p,p-1)$, both of modular freedom $10p+8$, in agreement with Enriques' result.

For regular surfaces $F$ with $d=2$ and $s>0$, an exceptional representation on a projective double ruled surface is no longer possible, but such representation on a double referable surface with conics as generators can be obtained. The classification and construction proceeds in much the same way as for $s=0$. The $s$ double curves $2C_i$ of $|C|$ are associated with $s$ isolated coincidence points of the involution of pairs of points of $F$ which generates the double surface, these coincidence points being base points of the relevant system of bisecant curves of $|C|$. Further it is shown that on any surface $F$ for which $s>0$ every divisor of zero is equivalent to a virtual curve of the type $\sum_1^{2h}C_i-hC$ $(2h\leqq s)$, from which it follows that the Severi invariant $\sigma$ of $F$ is given by $\sigma=2^{s-1}=2^{P_2-2p}$.

Finally it is shown that the surfaces of the family $\mathfrak{C}_1(p)$ which possess an infinite discontinuous group of self-transformations separate themselves into an enumerable infinity of irreducible algebraic systems of modular freedom $9p+8$. This leads to an analysis of the conditions under which the minimum absolute base number 2 of a generic surface of $\mathfrak{C}_1(p)$ can be exceeded for special members of the family. *J. G. Semple* (London).

**Nash, John. Real algebraic manifolds.** Ann. of Math. (2) 56, 405–421 (1952).

An algebraic variety defined over the real field in Euclidean $n$-space $E_n$ may consist of several sheets. Any one of these which has no singular points is called an algebraic sheet. If $M_d$ is a real closed differentiable manifold, it is known that it can be embedded differentiably in $E_n$, for a suitable value of $n$, and the principal theorem proved in this paper shows that this embedding can be approximated by an algebraic sheet $M_d'$ belonging to an algebraic variety $V_d$. If the other sheets of $V_d$ do not meet this sheet, $M_d'$ is said to be a proper representation of $M_d$, and it is shown that a proper representation of $M_d$ exists in $E_n$ if $n\geqq 2d+1$.

Let $M_d$ be any real closed analytic manifold and suppose that on it there exists a ring $R$ of analytic functions such that (a) in $R$ there exists a set of functions which define an algebraic representation of $M_d$, (b) if $f_1,\cdots,f_{d+1}\in R$, there exists a real polynomial $\phi(X_1,\cdots,X_{d+1})$ such that $\phi(f_1,\cdots,f_{d+1})=0$ on $M_d$, (c) $R$ is not contained in any larger ring having these properties. The pair $(M_d,R)$ is called a real algebraic manifold, and two manifolds $(M_d,R)$, $(M_d',R)$ are said to be equivalent if there exists an isomorphism between $R$ and $R'$. The second part of the paper develops a few properties of real algebraic manifolds, most of which correspond to well-known properties of algebraic geometry. The most important result obtained is that two real algebraic manifolds are equivalent if and only if they are analytically homeomorphic. *W. V. D. Hodge.*

**Andreotti, Aldo. Sopra il problema dell'uniformizzazione per alcune classi di superficie algebriche.** Rend. Accad. Naz. dei XL (4) 2, 111–127 (1951).

Démonstration complète du théorème suivant énoncé par E. Picard [Picard et Simart, Théorie des fonctions algé-

briques de deux variables indépendantes, t. II, Gauthier-Villars, Paris, 1906, p. 465]: Soit $f(X)=0$ l'équation d'une surface algébrique $F$ de l'espace ordinaire $S_3$ telle que: (1) il existe une transformation rationnelle de $F$ en elle-même $\tau$: $X'=R(X)$; (2) $\tau$ possède un point uni en un point simple $O$ de $F$; si $xOy$ est le plan tangent à $F$ en $O$, pour un choix convenable des axes de coordonnées, au voisinage de $O$, $\tau$ est définie au 2ème ordre près par $x'=ax$, $y'=by$; (3) $|a|>1$, $|b|>1$ et $a^h\neq b$, $b^h\neq a$ pour tout entier $h\geqq 2$. Alors, il existe une application du plan des $u$, $v$ complexes dans $S_3$: $(u,v)\to\Phi(u,v)$ définie par trois fonctions méromorphes, uniformes, deux à deux fonctionnellement indépendantes et satisfaisant aux équations fonctionnelles: $f(\Phi(u,v))=0$ et $\Phi(au,bv)=R(\Phi(u,v))$; $\Phi(u,v)$ définit une représentation paramétrique globale de $F$ (uniformisation).

Les transformations rationnelles d'une surface irrationnelle de genre linéaire $>1$ en elle-même étant birationnelles et en nombre fini, les surfaces irrationnelles uniformisables par des $\Phi(u,v)$ ont le genre linéaire $\leqq 1$, donc appartiennent à l'une des classes suivantes: (a) surfaces elliptiques, surfaces de Picard, et surfaces référables à des réglées qu'on sait uniformiser par d'autres moyens; (b) surfaces ayant un faisceau de courbes elliptiques de genre égal à l'irrégularité; (c) surfaces ayant tous les genres égaux à 1. L'auteur donne les premiers exemples connus de surfaces uniformisables par des $\Phi(u,v)$ appartenant aux classes (b) et (c): surface générique telle que $p_g=0$, $P_2=1$, $P_3=0$ pour la classe (b); surface générique du 4ème ordre de $S_3$ contenant un couple de droites sécantes (resp. de droites gauches) pour la classe (c). *P. Dolbeault* (Paris).

**Andreotti, Aldo. Sopra le varietà di Picard di una superficie algebrica.** Rend. Accad. Naz. dei XL (4) 2, 129–137 (1951).

A une surface algébrique $F$ d'irrégularité $q$, on associe deux variétés de Picard de dimension complexe $q$: l'une $V$ est la variété abélienne définie par la matrice de Riemann $\Omega$ des périodes des intégrales simples de première espèce de $F$, l'autre $V'$ est la variété abélienne image des systèmes linéaires d'un système continu $\infty^q$ de $F$ [Castelnuovo, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **14**, 545–556, 593–598, 655–663 (1905)]. Alors: (1) $V'$ est la variété image des systèmes continus de systèmes linéaires d'hypersurfaces sur la variété $V$; (2) si $V$ et $V'$ sont deux variétés abéliennes vérifiant (1), elles sont birationnellement distinctes sauf si (a) les diviseurs de la matrice $\Omega$ sont tous égaux à 1 ou (b) la matrice $\Omega$ admet la multiplication complexe. De plus, $V'$ est représentée sur $V$ au moyen d'une involution. On établit d'abord (2) à l'aide de la théorie des fonctions thêta sur les variétés abéliennes puis (1) à l'aide d'un critère d'équivalence linéaire pour les diviseurs de $F$. (1) est étendu à une variété algébrique $M$ de dimension quelconque à partir des résultats classiques suivants: la variété $V(M)$ (resp. $V'(M)$) est la variété $V$ (resp. $V'$) d'une surface $F$, section plane générique de $M$. *P. Dolbeault.*

## *Differential Geometry*

**Sorace, O. Su di una particolare curva di cui siano assegnate la curvatura e la torsione in funzione dell'arco.** Rend. Accad. Sci. Fis. Mat. Napoli (4) 14 (1946–47), 155–166 (1948).

La curva di cui siano assegnate come curvatura e torsione rispettivamente le funzioni dell'arco: $1/\rho(t)=a$ sen $ct$, $1/\tau(t)=a\cos ct$, viene trovata mediante un'equazione integrale di Volterra di seconda specie. Vengono studiate le proprietà di questi nuovo curva. *Author's summary.*

**Noto, Silvia. Sulle equazioni differenziali del tipo (Q).** Boll. Un. Mat. Ital. (3) 7, 298–301 (1952).

Let $F$ be a four-parameter family of curves on a surface with coordinates $u$, $v$. The curves which contain an arbitrary lineal element $E_1$: $(u_0, v_0, v_0')$ and pass through an arbitrary point $B$: $(u_1, v_1)$ form a one-parameter subfamily $F_1$. If, for every choice of $E_1$ and $B$, there is a projective relation between the second-order elements of the curves of $F_1$ at $E_1$ and the tangents of the curves of $F_1$ at $B$, the author says that $F$ has the "projective property". The condition for this is the identical vanishing of a certain expression $\Phi$. In general, when $\Phi$ is expanded in powers of $u_1-u_0$, the expansion begins with a term in $(u_1-u_0)^{19}$. The family $F$ is said to possess the projective property in at least the first, second, or third approximation when the degree of the first term is $>19$, $>20$, or $>21$, respectively. It is shown that if $F$ possesses the projective property in at least the first approximation, the defining differential equation is of the form

$$v^{iv}=A(u,v,v',v'')v'''^2+B(u,v,v',v'')v'''+C(u,v,v',v'').$$

If $F$ possesses the projective property in at least the second approximation, the function $A(u,v,v',v'')$ is of the form $4[3v''+\rho(u,v,v')]^{-1}$. The case in which $F$ possesses the projective property in at least the third approximation, which leads to complicated formulae, is also considered in the paper. *L. A. MacColl* (New York, N. Y.).

**Rangaswami Aiyer, K. Differential geometry of the cylindroid.** J. Annamalai Univ. 17, 91–93 (1952).

**Mishra, R. S. On isotropic congruences.** Ganita 2, 45–49 (1951).

L'auteur reprend, par le calcul tensoriel, l'étude des congruences rectilignes isotropes. Il étudie, en particulier, les relations entre le facteur de proportionnalité des deux formes quadratiques de Sannia d'une telle congruence et la fonction caractéristique de Bianchi, et retrouve quelques autres résultats sur les représentations sphériques des surfaces moyenne et enveloppée moyenne de la congruence. *P. Vincensini* (Marseille).

**Berezina, L. Ya. On the middle envelope of a congruence of normals.** Uspehi Matem. Nauk (N.S.) 7, no. 3(49), 121–122 (1952). (Russian)

For a normal congruence the distance between the points of tangency $P$ of the middle plane at its envelope and the central point of a ray is equal to the product of half the focal distance and the tangent of the angle between the ray and the normal to the middle surface. Moreover, the plane through the ray and $P$, and the plane through the ray and the line in which intersect the middle plane and the tangent plane at the middle surface, lie symmetrically with respect to the bisector of the focal planes. The proof is given by means of the trihedron of S. D. Rossinskiĭ [C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 54–56 (1943); these Rev. 6, 104] and an (unavailable) article by the author [Latvijas PSR Zinātņu Akad. Vēstis 1951, no. 8(49)]. *D. J. Struik.*

**Berezina, L. Ya. On a pair of applicable surfaces with constant distance between corresponding points.** Uspehi Matem. Nauk (N.S.) 7, no. 3(49), 123–124 (1952). (Russian)

On both sides from the central point $A$ on a ray $r$ of an isotropic congruence equal segments $AM_1=AM_2$ are set

out. The surfaces formed by $M_1$ and $M_2$ are applicable [see Finikov, Theory of congruences, Moscow-Leningrad, 1950, p. 39; these Rev. 12, 744]. Let now $i'$ be the angle between $r$ and the normal $n_k$, $k=1, 2$, to the surface $M_k$ at $M_k$; $i$ the angle between $r$ and the normal to the middle surface; $\alpha$ the angle between the plane through $r$ and the normal to the middle surface, and the plane through $r$ and $n_k$; then $\tan i'=\tan i \cos \alpha$. The $i'$ and $\alpha$ are the same for $M_1$ and $M_2$ using the trihedron of S. Rossinskiĭ: $\alpha_1=-\alpha_2=\alpha$, $i_1=i_2=i'$. See the literature cited in the preceding review.

*D. J. Struik* (Cambridge, Mass.).

**Berezina, L. Ya. Some relations concerning bilaterally stratified pairs of congruences.** Doklady Akad. Nauk SSSR (N.S.) **86**, 5–6 (1952). (Russian)

Four independent algebraical relations are derived, which hold for a general bilaterally stratified pair. These relations, together with a theorem of Finikov [Mat. Sbornik N.S. 12(54), 287–314 (1943); these Rev. 6, 19], are sufficient for the determination of such a pair. The case of specially symmetric pairs is excluded.

*D. J. Struik.*

**Havlíček, Karel. Surfaces réglées qui sont enveloppes de sphères.** Czechoslovack Math. J. **1(76)** (1951), 187–197 (1952) = Čehoslovack. Mat. Ž. **1(76)** (1951), 213–224 (1952).

The following theorem is proved: Given a real surface which envelopes a family of spheres depending on one parameter, the necessary and sufficient condition that this be a ruled surface (with real generators) is that this surface be a ruled surface of revolution, i.e., a cylinder of revolution, a cone of revolution, or a ruled hyperboloid of revolution.

*C. B. Allendoerfer* (Seattle, Wash.).

**Wunderlich, Walter. Über die $L$-Torsen der Flächen 2. Klasse.** Arch. Math. **3**, 44–49 (1952).

The osculating spheres of a developable surface $\Lambda$ are the spheres that touch four consecutive tangent planes of $\Lambda$. The $L$-developables of a surface $\Phi$ are those $\Lambda$'s whose generators are tangents of $\Phi$ and whose osculating spheres are tangent spheres of $\Phi$. Let $\Phi$ be a regular quadric. Then $\Lambda$ is an $L$-developable of $\Phi$ if and only if 1) its generators are common tangents of $\Phi$ and of a confocal quadric $\Psi$ and 2) its edge of regression lies on $\Psi$. The latter curve then is a geodesic on $\Psi$. The locus of the centers of the osculating spheres of $\Lambda$ is an affine image of the curve $\Lambda\cap\Phi$. The author also discusses the case that the surface $\Phi$ of class two degenerates into a conic, in particular, into a circle. He finally points out that the $L$-developables of certain surfaces of class four are Laguerre transforms of those of surfaces of class two.

*P. Scherk* (Los Angeles, Calif.).

**Müller, Hans Robert. Über die Hüllkurven monofokaler Kegelschnitte.** Math. Nachr. **7**, 289–292 (1952).

Die Hüllkurven der Kegelschnitte mit dem Brennpunkte $F$, die eine feste Hauptachsenlänge besitzen und durch einen festen Punkt $P$ hindurchgehen, sind Kegelschnitte mit den Brennpunkten $F$ und $P$ [vgl., z.B., Strubecker, Math. Nachr. 4, 36–46 (1951); diese Rev. 12, 856]. Verf. gibt einen Beweis mittels räumlicher Betrachtungen, wobei die Monofokalen Kegelschnitte als ebene Schnitte von Drehkegeln aufgefasst werden.

*O. Bottema* (Delft).

**Müller, Hans Robert. Elliptisch-metrische Eigenschaften des Strahlgewindes.** Monatsh. Math. **56**, 96–100 (1952).

Metrische Eigenschaften des Strahlkomplexes in der elliptischen Geometrie. Die (zwei) Achsen des Gewindes. Erklärung des Gewindes als Gesamtheit der Bahnnormalen bei einer Schraubenbewegung des Raumes. [Anmerkung des Ref.: Die Resultate des Verf. sind nicht neu und können auf die nicht-Euclidische $N$-dimensionale Geometrie erweitert werden [vgl. Bottema, Nederl. Akad. Wetensch., Proc. 35, 507–521, 681–693 (1932)].]

*O. Bottema.*

**Müller, Hans Robert. Der Satz von Malus und Dupin bei elliptischer Metrik und seine kinematische Deutung.** Monatsh. Math. **56**, 144–149 (1952).

Beweis des Malus-Dupinschen Satzes im elliptischen Raum. Erweiterung für die Brechung eines beliebigen Strahlensystems mit Betracht auf einer von Cartan und Blaschke herrührenden Integralinvariante, welche für Normalensysteme verschwindet. Ein kinematisches Analogon.

*O. Bottema* (Delft).

**Blank, Ya. P. On translation surfaces of an elliptic space.** Učenye Zapiski Har'kov. Gos. Univ. **28**, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **20**, 61–76 (1950). (Russian)

The existence of two three-parameter subgroups of Clifford translations in the group of motions of elliptic space is used as a point of departure for the study of translation surfaces in elliptic space. The groups of translations are the right and left translations of the group $X^*=X\cdot A$ and $X^*=A\cdot X$ where $A$ and $X$ are quaternions. It was Chebotarev who generalized the idea of translation surfaces. Starting with an arbitrary Lie group one considers a surface carrying $\infty^1$ curves which are transformed into each other by this group (systems of imprimitivity). The results obtained are in general analogous to those for ordinary space although some questions are raised but not answered.

*M. S. Knebelman* (Pullman, Wash.).

**Strubecker, Karl. Äquiforme Geometrie der isotropen Ebene.** Arch. Math. **3**, 145–153 (1952).

Geometrie in einer reellen Ebene $(x_0, x_1, x_2)$ oder $(1, x, y)$ mit einer Ferngeraden $(x_0=0)$ und zwei zusammengefallenen isotropen Punkten $(0, 0, 1)$. Sie ist selbstdual. Die Entfernung der Punkte $(1, x_1, y_1)$ und $(1, x_2, y_2)$ ist $x_2-x_1$. Die Metrik ist relativ invariant gegenüber der viergliedrigen Gruppe $X=a_1+ax$, $Y=b_1+bx+ay$. Das Bogenelement ist $ds=dx$, die Krümmung der Kurve $y(x)$ ist $d^2y/dx^2$. Für die weitere Entwicklung der Differentialgeometrie werden duale Zahlen angewandt; ein Punkt der Ebene wird durch $z=x+\epsilon y$ $(\epsilon^2=0)$ dargestellt.

*O. Bottema* (Delft).

**Morduhaĭ-Boltovskoĭ, D. D. On the curvature of space curves in Lobačevskiĭ space.** Mat. Sbornik N.S. **30(72)**, 483–508 (1952). (Russian)

Two curvatures are considered for a curve $C$: $x(s)$ in three-dimensional hyperbolic space: the ratios $K_t$, $K_n$ of the angles between the tangents and normal planes of $C$ at $x$ and $x+dx$, respectively, to $ds$. The latter may not exist (if the planes are hyperparallel); in that case it may be interpreted as a purely imaginary number whose absolute value measures the ratio of the distance between the planes to $ds$. The two curvatures are related by $K_t^2=K_n^2+k^{-2}$, where $k$ is the space constant. The torsion $T$ and "complete curvature" $K_c$ are the ratios of the angle between the osculating planes and rectifying planes of $C$ at $x$ and $dx$, respectively, to $ds$. Then $K_c^2=K_t^2+T^2$. These formulas are first derived analytically and again by synthetic arguments. The paper then turns to properties of the surfaces formed by the tangents, binormals, principal normals, or axes of curva-

ture (=intersection of infinitesimally close normal planes). For instance, the given curve is an asymptotic line on its surface of principal normals. The locus of the centers of the osculating spheres (if they exist) is the edge of regression of the surface of curvature axes. Curves of constant curvature are discussed. Such a curve has common principal normals with the locus of the centers of the osculating spheres. *H. Busemann* (Los Angeles, Calif.).

**Teixidor, J. On hexagonal and octahedral four-webs of planes.** Collectanea Math. 4, 93–122 (1951). (Spanish)

Given an $(n+1)$-web $W$ of [$(n-1)$-dimensional] hyperplanes in $n$-space, any $n-2$ hyperplanes which belong to different families of $W$ will intersect in a [two-dimensional] plane. Suppose the three remaining families of $W$ always intersect such planes in hexagonal webs. Then the hyperplanes of $W$ are the osculating hyperplanes of an $n$-dimensional algebraic curve of class $n+1$. This generalizes theorems of Graf and Sauer [$n=2$; cf. Blaschke and Bol, Geometrie der Gewebe, Springer, Berlin, 1938, pp. 24 ff.] and Bol [$n=3$; Abh. Math. Sem. Univ. Hamburg **10**, 119–134 (1934)]. The present proof makes no differentiability assumptions. Furthermore, a simplified proof of Sauer's theorem on octahedral webs of planes in 3-space is given [Blaschke and Bol, pp. 47 ff.; cf. also Bartsch, Abh. Math. Sem. Univ. Hamburg **17**, 1–21 (1951); these Rev. **13**, 277]. One of the author's main tools is the "hexagonal web of points" obtained out of $W$ by means of a duality. In the last sections a congruence of skew curves of order four of the first kind is studied in detail. It consists of those curves through four given co-planar points no three of which are collinear that have given tangents at three of the points.

*P. Scherk* (Los Angeles, Calif.).

**Hartman, Philip, and Wintner, Aurel. On the theory of geodesic fields.** Amer. J. Math. 74, 626–644 (1952).

The usual theorems about geodesics on a surface require it to be of class $C^2$ or higher, but do not apply to surfaces of class $C^0$ or $C^1$. This paper considers for the latter cases the theorems of Gauss on orthogonal trajectories, the related theorem of Jacobi concerning multipliers, Riemann's invariant relations which are both necessary and sufficient for normal geodesic coordinates, and Beltrami's characterization of the non-Euclidean and spherical metrics as geodesic maps of the Euclidean geometry. *C. B. Allendoerfer.*

**Hartman, Philip, and Wintner, Aurel. On geodesic torsions and parabolic and asymptotic curves.** Amer. J. Math. 74, 607–625 (1952).

This paper continues the authors' critical examination of classical differential geometry [same J. **72**, 757–774 (1950); **73**, 149–172 (1951); these Rev. **12**, 357, 742], in which attention is paid to the assumptions of differentiability and to the vanishing of various quantities. Here attention is first directed at the notion of geodesic torsion on a surface of class $C^2$. It is noted that the usual definition fails, and the geodesic torsion $\gamma$ is here defined as det $(X', N, N')$.

Three theorems are proved: I. On a surface of class $C^2$, a geodesic arc of non-vanishing curvature possesses a torsion. II. If $\Gamma$ is a curve of class $C^1$ on a surface of class $C^2$, then $\gamma=0$ holds at every point of $\Gamma$ if and only if $\Gamma$ is a line of curvature. III (Beltrami-Enneper). If $\Gamma$ is an arc of class $C^1$ on a surface of class $C^2$, then $\gamma^2=-K$ at every point of $\Gamma$ if and only if every tangent vector of $\Gamma$ either is, or is orthogonal to, an asymptotic direction.

P. Franklin [J. Math. Physics **13**, 253–260 (1934)] asserted that a "regular" parabolic curve must be a line of curvature. His proof has been challenged; but it is shown here that although the proof can be saved, the definition of "regular" makes the result almost vacuous.

The authors clarify the relationship among the following properties: ($\alpha$) $\Gamma$ is an asymptotic curve; ($\beta$) $\Gamma$ is a line of curvature; ($\gamma$) $\Gamma$ is a parabolic curve; as follows: ($\alpha$) and ($\beta$) imply ($\gamma$); ($\alpha$) and ($\gamma$) imply ($\beta$); ($\beta$) and ($\gamma$) do not imply ($\alpha$); ($\alpha$), ($\beta$), and ($\gamma$) are all satisfied if and only if $\Gamma$ is a plane curve along which the tangent plane to the surface does not vary. A similar approach is made to a theorem of van Kampen on asymptotic curves [Amer. J. Math. **61**, 992–994 (1939); these Rev. **1**, 85], whose proof is false but which is indeed true itself.

Finally three more theorems are proved carefully: IV. Let $S$ be a surface, of class $C^2$, having no elliptic points, and let $\Gamma$ be a curve of class $C^2$ on $S$. Then $\Gamma$ is an asymptotic curve if and only if $\kappa(S)=|\beta(s)|$ where $\kappa(\geqq 0)$ is the curvature and $\beta$ the geodesic curvature of $\Gamma$. If, in addition, $\kappa(s)\neq 0$ holds on an asymptotic curve $\Gamma$, then $\tau(s)\equiv\gamma(s)$, where $\tau(s)$ is the torsion and $\gamma(s)$ the geodesic torsion of $\Gamma$. V (Bonnet). Under the above hypotheses let $\Gamma$ have, at some point $P$, an asymptotic direction and a non-vanishing curvature. Then $(3\tau_0-\tau)\kappa=\pm 2\tau_0\kappa_0$, where $\kappa$ and $\tau$ denote curvature and torsion, $\pm\kappa_0$ and $\tau_0$ the geodesic curvature and geodesic torsion of $\Gamma$ at $P$. VI (Beltrami). Let $P$ be a hyperbolic point of a surface $S$ of class $C^3$, and let $\Gamma$ denote a branch of the intersection $ST$, where $T$ is the plane tangent to $S$ at $P$. Then $\Gamma$ has at $P$ a (continuous) curvature $\kappa\geqq 0$, and $3\kappa=2\kappa_0$ holds for the curvature, $\kappa_0\geqq 0$, of the asymptotic tangent to $\Gamma$ at $P$. *C. B. Allendoerfer* (Seattle, Wash.).

**Sauer, Robert. Gruppen infinitesimaler Kollineationen.** S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 129–138 (1952).

If $A$ and $B$ are the matrices of two infinitesimal collineations in $P_n$, the matrices $\lambda A+\mu B$ define a group of infinitesimal collineations. It is shown that for $n=2$ the locus of the invariant points of these collineations is a curve $C_3$; the invariant lines are bisecants of $C_3$ and envelope a $C_3'$. Starting with three infinitesimal collineations $A$, $B$, and $C$ in $P_3$ one obtains the group $\lambda A+\mu B+\nu C$. The paper contains some properties of the locus of the invariant points, of the invariant lines, and of the invariant planes.

*J. Haantjes* (Leiden).

**Pan, T. K. Normal curvature of a vector field.** Amer. J. Math. 74, 955–966 (1952).

The first part of this paper concerns a field $v$ of vectors lying in a surface $S$ of ordinary Euclidean space. The derived vector $v$ along a curve $C$ of $S$ has a tangential component which, under the name of angular spread vector of $v$ along $C$, has been extensively studied by Graustein [Trans. Amer. Math. Soc. **34**, 557–593 (1932)]. The author in this paper considers the normal component which he names the normal curvature vector of the vector field $v$ with respect to the curve $C$. Many notions which are well known in the particular case when $v$ is the tangent vector to $C$ are generalized, such as normal curvature, asymptotic directions, asymptotic lines, lines of curvature, etc. A curve on the surface, along which the vectors of $v$ are tangent, is called a curve of the vector field. If such vectors are also parallel in the sense of Levi-Cività, the curve is then called an indicatrix of the vector field. In the last section of the paper, the results obtained for a vector field on a surface in ordinary space are

generalized to vector fields on any subspace of a Riemannian space. *E. T. Davies* (Southampton).

**Gołąb, St., et Wróbel, T. H. Courbure et torsion géodésique pour les courbes situées sur les hypersurfaces à $n-1$ dimensions plongées dans l'espace à $n$ dimensions.** Ann. Soc. Polon. Math. **24** (1951), 25–51 (1952).

As a continuation of an earlier paper of Gołąb [same Ann. **22**, 97–156 (1950); these Rev. **11**, 690], the authors consider a $V_{n-1}$ imbedded in an Euclidean $R_n$. Associated with a curve $C$ on $V_{n-1}$ choose an orthonormal ennuple $t_1, \cdots, t_n$ where $t_1$ is tangent to $C$ and $t_n$ is normal to $V_{n-1}$. Then it is proved that $\alpha_\lambda$, $\beta_\lambda$, $\gamma$ ($\lambda=2, \cdots, n$) exist such that: $Dt_1=\sum_\lambda \alpha_\lambda t_\lambda+\gamma t_n$; $Dt_\mu=-\alpha_\mu t_1+\beta_\mu t_n$ ($\mu=2, \cdots, n-1$); $Dt_n=-\gamma t_1-\sum_\lambda \beta_\lambda t_\lambda$. The quantities:

$$\alpha=(\textstyle\sum_\lambda \alpha_\lambda^2)^{1/2}, \quad \beta=(\sum_\lambda \beta_\lambda^2)^{1/2},$$

and $\gamma$ are independent of the choice of the ennuple. Call $\alpha$ the geodesic curvature, $\beta$ the geodesic torsion, $\gamma$ the normal curvature. Then $\alpha=0$ characterizes the geodesics, $\beta=0$ the lines of curvature, and $\gamma=0$ the asymptotic lines of the second order. *C. B. Allendoerfer* (Seattle, Wash.).

**Wróbel, T. H. L'interprétation géométrique de la courbure et de la torsion géodésique à l'aide de la représentation hypersphérique de la courbe.** Ann. Soc. Polon. Math. **24**(1951), 52–55 (1952).

With the notations of a previous paper [see the preceding review], let $\kappa_1$ be the first curvature of $C$; $C_1$ the hyperspherical image of $C$ corresponding to a mapping by $t_1$, $C_n$ the corresponding image obtained from $t_n$; $\omega$ the angle between $t_n$ and the principal normal to the curve; $\phi_1$ the angle between the tangent to $C_1$ and $t_n$; $\phi_n$ the angle between the tangent to $C_n$ and $t_1$. Then two theorems are proved: I) If $\gamma\neq 0$, then $\alpha=\kappa_1 \cos\omega \tan\phi_1$. II) If $\gamma\neq 0$, then $\beta=-\kappa_1\cos\omega\tan\phi_n$ (a generalization of a theorem of Knoblauch). *C. B. Allendoerfer* (Seattle, Wash.).

**Saban, Giacomo. Sulle curve sghembe in uno $S_n$.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 309–321 (1950).

The tangents of a curve $C$ in Euclidean $N$-space describe in an osculating hyperplane a curve $C^1$; $C$ and $C^1$ touch in the osculating point. If $P_i$ and $P_i^1$ are the $i$th radii of curvature of $C$ and $C^1$ respectively, then

$$P_i^1/P_i=(n-1)(n-i)/n(n-i-1).$$

The theorem is a generalization of Beltrami's [Nouvelles Ann. Math. (2) **4**, 258–267 (1865)] for $n=3$, $i=1$. Extension to a related theorem in projective space. *O. Bottema.*

**Bol, G. Zur projektiven Differentialgeometrie der Kurven in der Ebene und im Raum.** Arch. Math. **3**, 163–170 (1952).

Consider a curve $\mathbf{x}(t)$ in a projective flat three-space. Then it is not difficult to associate with each point $t$ a tetrahedron $T(t)$ intrinsically defined by means of the parameter $t$ and dependent on it. In his book Projektive Differentialgeometrie, I. Teil [Vandenhoeck & Ruprecht, Göttingen, 1950; these Rev. **11**, 539] the author initiated successfully a new method of dealing with projective differential geometry which in this case consists of finding the loci of elements of $T$ for all possible choices of parameter. In this short note he gives three applications of this method. Let $\mathbf{x}(t)$ be a plane curve, $C$ its osculating conic section at $t=0$. Consider the intersection $\mathbf{p}$ of the tangential line at $t=0$ with the line $\mathbf{x}(t)\mathbf{x}(-t)$. If $\mathbf{x}(t)$ is of class 3, then the limiting position of $p$ exists for $t=0$. Call it $\mathbf{t}$ and denote by $\mathbf{y}$ the intersection of $C$ with the polar line of $\mathbf{t}$ (with respect to $C$). Then we have a frame of reference $\mathbf{x}(0)$, $\mathbf{t}$, $\mathbf{y}$ at $t=0$, for which the Frenet formulae can easily be found. As $t$ changes, the point $\mathbf{y}$ describes the conic $C$. Similar results hold also for space curves and even for linear complexes. *V. Hlavatý* (Bloomington, Ind.).

**Barner, Martin. Zur projektiven Differentialgeometrie der Kurven des $n$-dimensionalen Raumes.** Arch. Math. **3**, 171–182 (1952).

Let $\mathbf{x}(t)$ be a curve in a flat $n$-dimensional projective space and

$$\text{(1)} \qquad \mathbf{x}_0=\mathbf{x}, \quad \mathbf{x}_1, \cdots, \mathbf{x}_n$$

its corresponding frame of reference with the "Frenet formulae"

$$\text{(2)} \qquad \mathbf{x}'_k=\mathbf{x}_{k+1}+\sum_{\mu=1}^{k} a_{k\mu}e_{\mu+1}\mathbf{x}_{k-\mu}, \quad \mathbf{x}_i=0 \quad \text{for} \quad i<0, \quad i>n,$$

where $e_\mu=e_\mu(t)$ and $a_{k\mu}$ are well-defined numerical constants. (For the case of a curved projective space see Hlavatý, Trudy Sem. Vektor. Tenzor. Analizu **2–3**, 119–150 (1935).) Every change of parameter has to be performed together with such a change of factor of proportionality so that the determinant of (1) remains constant. This combined transformation leads to

$$x_k^*=\sum_{\nu=0}^{k} F_{k\nu}\mathbf{x}_{k-\nu}$$

where $F_{k\nu}$ are well-defined functions derived from the parameter transformation. The locus of $x_n^*$ is a curve $C_n$ termed an "harmonical-$C_n$". The paper is devoted to the proof of the following statement: There is a unique curve $C$ which has with $\mathbf{x}(t)$ all osculating spaces in common in $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ and goes through $\mathbf{x}(t_1)$, $\mathbf{x}(t_2)$, and $\mathbf{x}(t_3)$. For $t_1\to t_2\to t_3$ this curve is the "harmonical-$C_n$". The proof is based on (2), on Rolle's theorem, and on two lemmas (very handy also for other purposes) involving the derivatives of the faces $\mathbf{X}_m$ of the frame (1) as well as the derivatives of the points $\mathbf{y}=\sum_{\alpha,\beta}^{0,\cdots,n}\gamma_{\alpha\beta}t^\beta\mathbf{x}_\alpha$ of $C$. *V. Hlavatý.*

**Vesentini, Edoardo. Sulle omografie definite da certe coppie di elementi differenziali tangenti.** Ricerca, Napoli **3**, no. 2, 26–28 (1952).

In a linear $n$-dimensional space ($n\geqq 2$), $S$, where $(x_1, x_2, \cdots, x_n)$ are projective non-homogeneous coordinates, two curves

$$L{:}\ \ x_r=a_rx_1^r+\cdots, \qquad L'{:}\ \ x_r=a_r'x_1^r+\cdots \qquad (a_ra_r'\neq 0,\ r=2, \cdots, n)$$

having in common a point $O(0, 0, \cdots, 0)$ and all the osculating spaces at $O$, admit the $n-1$ projective invariants $I_r=a_r/a_r'$ [cf. B. Segre, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **22**, 392–399 (1935)]. Here the author expresses $I_r$ by means of the roots of the characteristic equation of any homography of $S$ in itself, which transforms the neighbourhood of order $n$ of $O$ upon $L$ into the neighbourhood of order $n$ of $O$ upon $L'$. *B. Segre* (Rome).

**Bompiani, Enrico. Sulla curvatura pangeodetica di una curva di una superficie dello spazio proiettivo.** Boll. Un. Mat. Ital. (3) **7**, 103–106 (1952).

Consider a surface (neither developable nor quadratic) in ordinary projective space. A curve on this surface has an extremal curvature $1/\rho$ and a pangeodesic curvature $1/k$,

as defined by Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 2, 466–470 (1925)]. Here it is proved that $1/2\rho^2=1/k^2$. *C. B. Allendoerfer* (Seattle, Wash.).

**Kanitani, Joyo. Sur la transformation infinitesimale du groupe d'holonomie.** Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 75–93 (1952).

A projective connection fixes a projective transformation of projective local space along any curve. The matrix of this transformation can be dealt with in several ways, for instance, by using Schlesinger's product integrals. Here it is done by means of expansions in series. The rather long formulae—one of them takes two and a half pages—lead at last to an expression for the matrix belonging to an infinitesimal part of the curve, and this expression contains a quantity $\chi^\alpha{}_{\beta ij}$; $\alpha, \beta=0, 1, \cdots, n$; $i, j=1, \cdots, n$, that can be expanded in a series containing the curvature tensor and its derivatives. By means of this result, it is possible to determine in §7 a projection connection with a given group of holonomy. In §8 the case is considered where a hyperplane is left invariant by the group of holonomy. Cartan [Leçons sur la théorie des espaces à connexion projective, Gauthier-Villars, Paris, 1937, p. 292] proved that a projective connection is affine if its group of holonomy is affine and that the connection is non-euclidean if this group leaves invariant one regular hyperquadric. These results are proved in §§8 and 9. *J. A. Schouten* (Epe).

**Tachibana, Syun-ichi. On generalized logarithmic spirals in Riemann spaces.** Nat. Sci. Rep. Ochanomizu Univ. 2, 28–30 (1951).

Consider a curve $C$ in a Riemann space $V_n$ whose tangent vector and $n-1$ normal vectors are denoted respectively by $\xi^i{}_{(1)}, \xi^i{}_{(2)}, \cdots, \xi^i{}_{(n)}$, and whose $i$th curvature is $\kappa_{(i)}$. Then $C$ is a generalized logarithmic spiral of order $p$ if: (i) $\kappa_{(p+1)}=0$, $p<n$; (ii) a point field $M(s)+\sum_{a=1}^{p+1}\alpha_a\xi^i{}_{(a)}e^i$ is independent of $s$ where it is assumed that $\alpha_b=\alpha c_{b-1}$, $c_{b-1}=\text{const.}\neq0$, $c_0=1$, $\alpha\neq0$, $M(s)$ is an arbitrary point of the curve, and $e^i$ are the vectors of a natural frame at $M$. The paper obtains formulae for the $\kappa_{(i)}$ of such a curve, and the differential equations of such curves of order 1.
*C. B. Allendoerfer* (Seattle, Wash.).

**Norden, A. P., and Cypkin, M. E. On a connection between ruled surfaces and curves of a Riemannian space.** Doklady Akad. Nauk SSSR (N.S.) 86, 23–26 (1952). (Russian)

The relation $v_\alpha x^\alpha=0$, $\alpha=1, 2, \cdots, 6$, defines the Plücker coordinates of a linear complex (or screw) $v$ in euclidean space $R_3$, if $x^\alpha$ are the coordinates of a line. The $x^1, x^2, x^3$ represent the unit vector in the direction of the line, the $x^4, x^5, x^6$ the moment. The invariants are given by $vw=v_1w_4+v_2w_5+\cdots+v_6w_1$, $v\bar{w}=\bar{v}w=v_1w_1+v_2w_2+v_3w_3$, where $\bar{v}=(0, 0, 0, v_1, v_2, v_3)$. Then $xx=0$ and $x\bar{x}=1$; $dxdx=2dld\varphi$, $dxd\bar{x}=d\varphi^2$, where $l$ is the shortest distance and $\varphi$ the angle of two lines. Now the complex is represented by a point in projective space $P_5$, a line (special complex) by a quadric $Q_4$ in $P_5$. On $Q_4$ curvilinear coordinates $u^i$, $i, j, k=1, 2, 3, 4$, are used to define a reference system $x, \bar{x}, x_i=\partial x/\partial u_i$. Then, if $p_{ij}=x_i\bar{x}_j$, $g_{ij}=x_ix_j$: $d\varphi^2=p_{ij}du^idu^j$, $2dld\varphi=g_{ij}du^idu^j$.

Since $p_{ij}$ is of rank 2 and $g_{ij}$ of rank 4 (and signature $+, +, -, -$), a Riemannian geometry can be defined on $Q_4$ with $d\sigma^2=g_{ij}du^idu^j$. Then a covariant differentiation can be defined with the formulas

$$\nabla_j x_i=-g_{ij}\bar{x}-p_{ij}x, \quad \nabla_j\bar{x}_i=-p_{ij}\bar{x}.$$

Then $\nabla_k g_{ij}=\nabla_k p_{ij}=0$. The curvature tensor is

$$R^r{}_{kji}=g_{i[k}p_{j]s}g^{rs}+p_{i[k}\delta_{j]}{}^r,$$

so that the geometry is symmetrical and conformally euclidean.

To every direction $v^i$ at a point $x$ on $Q_4$ lying in $Q_4$ belongs a screw $v=v^ix_i$, whose axis coincides with the striction normal (principal normal of the edge of regression) and whose pitch is equal to the parameter of distribution of the developable surface represented by the curve on $Q_4$ of which $v^i$ is the tangent direction. The screw $Dv/d\sigma=x_i\delta v^i/\delta\sigma$ is called the derivative of the direction screw $v$, $\delta$ represents the covariant derivative with respect to $g_{ij}$.

A curve $u^i=u^i(s)$ on $Q_4$ represents a ruled surface in $R_3$, $x=x(s)$, $s$ is the arc length in $R$. Then Frenet formulas can be derived:

$$\underset{0}{t}=dx/ds,\quad \frac{D}{ds}\underset{0}{t}=\underset{0}{\epsilon}\underset{1}{\kappa}\underset{1}{t},\quad \frac{D}{ds}\underset{1}{t}=\underset{1}{\epsilon}(-\underset{1}{\kappa}\underset{0}{t}+\underset{2}{\kappa}\underset{2}{t}),$$

$$\frac{D}{ds}\underset{2}{t}=\underset{2}{\epsilon}(-\underset{2}{\kappa}\underset{1}{t}+\underset{3}{\kappa}\underset{3}{t}),\quad \frac{D}{ds}\underset{3}{t}=-\underset{3}{\epsilon}\underset{2}{\kappa}\underset{2}{t},$$

where $\underset{i}{t}\underset{i}{t}=\underset{i}{\epsilon}=\pm1$, $\underset{i}{t}\underset{j}{t}=0$ $(i\neq j)$, except in the "isotropic" cases where $\underset{0}{t}\underset{0}{t}=0$, or $\underset{1}{t}\underset{1}{t}=0$, or $\underset{2}{t}\underset{2}{t}=0$.

Classification of curves on $Q_4$ can now be used to classify ruled surfaces $S$ in $R$. Here are some examples. Necessary and sufficient condition that a) $\underset{1}{\kappa}=0$, is that $S$ be helicoidal; b) $\underset{2}{\kappa}=0$ and parameter of distribution be constant, is that $S$ be generated by the binormals of a curve of constant torsion; c) $\underset{2}{\kappa}=0$ and parameter of distribution be not constant, is that $S$ be generated by the striction normals of a developable surface, the edge of regression of which is either a Bertrand curve or a helix; d) $\underset{0}{t}\underset{0}{t}=0$ (zero-isotropic), is that $S$ be developable. *D. J. Struik*.

**Ōtsuki, Tominosuke. On the spaces with normal projective connexions and some imbedding problem of Riemannian spaces. II.** Math. J. Okayama Univ. 2, 21–40 (1952).

[For part I see same J. 1, 69–98 (1952); these Rev. 13, 985.] The author investigates the conditions under which a given Riemannian space $V_n$ of dimension $n$ can be imbedded as a hypersurface in a Riemannian space $V_{n+1}$ satisfying the following conditions: 1) The group of holonomy of the space with a normal projective connection corresponding to $V_{n+1}$ fixes a hyperquadric and $V_n$ is its image in $V_{n+1}$; 2) if $y$ is a scalar such that the hypersurface is given by $y=0$, the orthogonal trajectories of the family of hypersurfaces on which $y$ is constant are geodesics in $V_{n+1}$. Necessary and sufficient conditions are established. If $n>2$ and if a further condition is satisfied, $V_n$ is either an Einstein space or its Ricci tensor satisfies the conditions

$$R^b{}_{a,c}=0,\quad R_a{}^\lambda R_\lambda{}^b=\frac{R}{n-1}R_a{}^b.$$

*S. Chern* (Chicago, Ill.).

**Chern, Shiing-shen, and Kuiper, Nicolaas H. Some theorems on the isometric imbedding of compact Riemann manifolds in euclidean space.** Ann. of Math. (2) 56, 422–430 (1952).

Let $M$ be a compact Riemann manifold which is imbedded in a Euclidean space $E$ of dimension $n+N$. Then it is proved

that $M$ has at least one point at which there is no real asymptotic directions. The second fundamental forms of $M$ are denoted by $A_{rij}$, and the forms $\phi_r$ are defined by $\phi_r = A_{rij}\omega_i\omega_j$. Let $\nu(P)$ be an integer such that $n-\nu(P)$ is the minimum number of linearly independent linear differential forms in terms of which $\phi_r$ can be expressed. $\nu(P)$ is called the index of relative nullity. Also let $\mu(P)$ be such that $n-\mu(P)$ is the minimum number of linearly independent linear differential forms in which $\Omega_{ij}$ (the curvature forms of $M$) can be expressed. $\mu(P)$ is called the index of nullity. It is proved that $\nu(P) \leqq \mu(P) \leqq N+\nu(P)$.

Now suppose that $M$ is given abstractly, so that only $\mu(P)$ is defined. It is proved that if $M$ has an index of nullity everywhere $\geqq \mu_0$, it cannot be isometrically imbedded in a Euclidean space of dimension $n+\mu_0+1$. This is an extension of a theorem of Tompkins [Duke Math. J. 5, 58–61 (1939)]. Also suppose that $M$ has a $q$-dimensional ($q=2$ or 3) linear subspace in the tangent space along whose plane elements the sectional curvatures are non-positive. Then $M$ cannot be isometrically imbedded in a Euclidean space of dimension $n+q-1$. Compare S. B. Myers for $q=2$ [Trans. Amer. Math. Soc. 71, 211–217 (1951), p. 215; these Rev. 13, 492].

Finally, suppose that $\mu$ is constant in a neighborhood of $M$. Then $M$ can be locally sliced into submanifolds of dimension $\mu$ which are everywhere tangent to the spaces of nullity and are locally flat in the induced metric.

*C. B. Allendoerfer* (Seattle, Wash.).

**Yano, Kentaro. On groups of homothetic transformations in Riemannian spaces.** J. Indian Math. Soc. (N.S.) 15 (1951), 105–117 (1951).

Verf. leitet eine Reihe von Sätzen über homothetische Transformationsgruppen in Riemannschen Räumen $V_n$ unter Benützung von Lieschen Differentialen ab. Die Sätze 1–3 des §2 beziehen sich auf die Gestalt des Fundamentaltensors desjenigen $V_n$ der eine homothetische Transformationsgruppe gestattet. Es wird ferner gezeigt, dass für die Existenz einer homothetischen Transformationsgruppe notwendig und hinreichend ist, dass dieselbe gleichzeitig konform und affin, oder konform und projektiv ist. Wird dieses Ergebnis analytisch ausgedrückt, dann erhält man für den die infinitesimale homothetische Transformationen bestimmenden Vektor $\xi^i$ und seine kovariante Ableitung ein gemischtes System von partiellen Differentialgleichungen. Die notwendigen und hinreichenden Bedingungen für das Vorhandensein einer homothetischen Transformation ergeben sich dann aus der algebraischen Verträglichkeit einer Kette von Integrabilitätsbedingungen den zuvor erwähnten Gleichungen. Es ergibt sich hieraus sofort, dass das Vorhandensein einer homothetischen Transformationsgruppe von maximaler Parameterzahl $\frac{1}{2}n(n+1)+1$ die euklidischen Räume charakterisiert. Die meisten Sätze dieses Paragraphen sind wie Verf. selbst bemerkt aus wesentlich anderem Wege von E. B. Shanks [Duke Math. J. 17, 299–311 (1950); diese Rev. 12, 359] bewiesen worden. In §3 bestimmt Verf. die Gestalt des Bogenelementes derjenigen $V_n$ die eine solche homothetische Transformationsgruppe besitzen, deren Bahnkurven geodätische Linien sind. Hier wird weiters nachgewiesen, dass eine Transformation, die geodätische Linien und konforme Kreise wieder in ebensolche Linien überführt, notwendig homothetisch ist. Den Abschluss dieses Paragraphen bildet der Satz, dass ein Einsteinscher Raum mit nichtverschwindendem Krümmungsskalar keine homothetische Transformationsgruppe besitzen kann. In §4 wird gezeigt, wie man aus der Lieschen Symbolen einer homothetischen Transformationsgruppe diejenigen einer Bewegungsgruppe herleiten kann. In §5 werden solche $V_n$ untersucht die zu einem gegebenen $V_n$ mit homothetischer Transformationsgruppe konform sind und entweder eine homothetische, oder eine Bewegungsgruppe besitzen. §6 beschäftigt sich mit der Frage, wann ein $V_n$ mit konformer bzw. affiner Transformationsgruppe eine homothetische Untergruppe besitzt. In §9 wird untersucht, wann es bei Vorgabe einer Transformationsgruppe deren Strukturkonstanten gewissen Bedingungen genügen, Riemannsche Räume gibt, für die diese Gruppe eine homothetische Transformationsgruppe ist.

*O. Varga* (Debrecen).

**Matsumoto, Makoto. Affinely connected spaces of class one.** Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 26, 235–249 (1951).

A space of affine connection $V^m$ is said to be of class one if it can be imbedded in an affine space of dimension $m+1$. The author obtains an algebraic characterization for such a space in a form similar to that of T. Y. Thomas [Acta Math. 67, 169–211 (1936)] for a Riemann space of class one.

*C. B. Allendoerfer* (Seattle, Wash.).

**Matsumoto, Makoto. The class number of embedding of the space with projective connection.** J. Math. Soc. Japan 4, 37–58 (1952).

A space with projective connection $P^m$ is said to be of class one if it can be imbedded in a projective space of dimension $m+1$. As in the paper reviewed above, the author obtains an algebraic characterization for such a space.

*C. B. Allendoerfer* (Seattle, Wash.).

**Bompiani, E. Sulle connessioni affini non-posizionali.** Arch. Math. 3, 183–186 (1952).

The author gives some discussions of the basic concepts of non-positional affine connections and shows that the connection in a Finsler space recently introduced by H. Rund [Math. Z. 54, 115–128 (1951); these Rev. 13, 159] can be obtained as a particular case.

*S. Chern.*

**Suguri, Tsuneo. Theory of invariants in the geometry of paths. I. Determination of covariant differentiations.** Proc. Japan Acad. 26, nos. 2–5, 97–103 (1950).

**Suguri, Tsuneo. Theory of invariants in the geometry of paths. II. Equivalence problems.** Proc. Japan Acad. 26, nos. 2–5, 104–106 (1950).

In the first paper an investigation is made of the theory of invariants in the geometry of paths under the "generalized rheonomic group", $\bar{x}^i = \bar{x}^i(x, t)$, $\bar{t} = t(t)$. This group contains all groups usually considered in geometries of paths, because generally, either $t$ is not transformed, or $t$ does not enter into the transformations of the $x$'s, or both. Starting from the equations for the paths:

$$x^{(m)i} + H^i(t, x, x^{(1)}, \cdots, x^{(m-1)}) = 0,$$

the author builds up sets of useful quantities, and then constructs a linear connection. The second paper gives conditions for the equivalence of two systems of paths in terms of the curvature and torsion quantities.

*A. Nijenhuis.*

**Perrin, F. The geometry of spinors.** Report of an International Conference on Elementary Particles, Bombay, 1950, pp. 17–22. The International Union of Pure and Applied Physics, Bombay, 1952.

The author gives a geometrical interpretation of two-component spinors in terms of the rotations of Euclidean three-space.

*A. H. Taub* (Urbana, Ill.).

**Milner, S. R. The relation of Eddington's $E$-numbers to the tensor calculus. I. The matrix form of $E$-numbers.** Proc. Roy. Soc. London. Ser. A. 214, 292–311 (1952).

**Milner, S. R. The relation of Eddington's $E$-numbers to the tensor calculus. II. An extension of tensor transformation theory.** Proc. Roy. Soc. London. Ser. A. 214, 312–329 (1952).

The author fixes a choice of the coordinate system in the four-dimensional complex space in which the Eddington $E$-numbers operate and also fixes a coordinate system in space-time, the four-dimensional real space in which the coefficients of an expansion of an $E$-number in terms of a basic set are considered as tensors. He then identifies the two four-dimensional spaces. The first paper is concerned with the calculus for transforming a second-order tensor in space-time into an $E$-number and the converse of this operation, and special equations are derived which hold in the restricted frames of references mentioned above. The second paper discusses the transformation induced on the $E$-number representation of a second-order tensor when this tensor is subjected to a tensor transformation. The author states that the theory propounded in these papers has physical applications and that these will be published in later papers.

Since the results of these papers are dependent on the choice of coordinate systems in two spaces and the identification of these spaces, it would seem that any physical applications of the results would have to be very special ones or be contained in a theory freed from these restrictions. Such a theory is the four-component spinor theory in which a distinction is made between the two spaces and in which a method exists for correlating coordinate and geometrical transformations in both spaces. Problems similar to those dealt with in these papers have been treated in spinor theory. *A. H. Taub* (Urbana, Ill.).

***Kawada, Yukiyoshi. Bibunshiki ron—Grassmann daisū to Lie gun. [Theory of differential forms—Grassmann algebras and Lie groups.]** Kawade-shobô, Tokyo, 1951. vi+194 pp. 300 Yen.

This book is intended as an introduction to the theory of differential forms of E. Cartan and its various applications. An outline of the book is as follows: In the first chapter, the author defines Grassmann algebras and gives some fundamental properties of such algebras. In chapter 2, differential forms are introduced together with formal rules of calculating those forms, and the theorem of Green-Stokes is proved for general algebraic complexes. Applying this, the author then proves in chapter 3 de Rham's theorem on differentiable manifolds. For simplicity, the proof is carried out only for 2-dimensional manifolds. In chapter 4, applications of differential forms to the theory of total differential equations and partial differential equations are discussed. In the last chapter the author gives an exposition of the theory of local Lie groups using the method developed earlier in the book. The theory of transformation groups and the theory of invariant differential forms on connected compact Lie groups are also discussed, showing, in particular, the method of determining the Poincaré polynomials of such Lie groups by means of their adjoint representations. The proofs mostly follow the usual lines, but the material is carefully arranged, so that the reader can understand it with as little preparatory knowledge as possible. *K. Iwasawa.*

## NUMERICAL AND GRAPHICAL METHODS

***Bickley, W. G., Comrie, L. J., Miller, J. C. P., Sadler, D. H., and Thompson, A. J. Bessel functions. Part II. Functions of positive integer order.** British Association for the Advancement of Science, Mathematical Tables, vol. X. University Press, Cambridge, 1952. xl+255 pp. £3; $11.00.

This long-awaited and valuable volume, Part II of the tables of Bessel functions prepared under the auspices of the British Association for the Advancement of Science, contains the following tables:

$J_n(x)$, $n=2(1)20$, $x=[0(.1 \text{ or } .01)10(.1)25;\ 8D]$, with $\delta^2$ or $\delta_m{}^2$;
$Y_n(x)$ or $y_n(x)$, $n=2(1)20$, $x=[0(.1 \text{ or } .01)10(.1)25;\ 8S]$, with $\delta^2$ or $\delta_m{}^2$;
$i_n(x)$ or $e^{-x}I_n(x)$, $k_n(x)$ or $e^xK_n(x)$, $n=2(1)20$, $x=[0(.1 \text{ or } .01)10(.1)20;\ 8S]$ with $\delta^2$ or $\delta_m{}^2$;
$J_n(x)$, $n=0(1)20$, $x=[.1(.1)25;\ 10D]$;
$Y_n(x)$, $n=0(1)20$, $x=[.1(.1)25;\ 10S]$;
$I_n(x)$, $K_n(x)$, $n=0(1)20$, $x=[.1(.1)20;\ 10S]$.

One is surprised not to find tables of zeros of the functions, since such tables for $J_n(x)$ were omitted from the last Harvard volume because their computations had been requested by the British Committee.

Throughout the typographical displays are admirable.

In the early pages of the volume an especially valuable section on "Functions and Formulae" gives a comprehensive list of formulae connected with the functions tabulated. Other sections deal with various problems of "Interpolation", "Preparation of the Tables", and "Bibliography". The last section, by J. C. P. Miller, "is concerned mainly with works that supplement the present volume". Thus the Harvard tables, and those of Badellino, of Faddeeva and Gavurin, and Lyusternik and Akušskiĭ, are samples of more than a score of entries.

This volume, of great practical importance, is the last of the fine series of British Association Mathematical Tables, but has been published under the auspices of the Royal Society. *R. C. Archibald* (Providence, R. I.).

***Fröberg, Carl-Erik. Hexadecimal conversion tables.** C. W. K. Gleerup, Lund, 1952. 20 pp. 3 kr.

In most automatic computing machines numerical calculations are carried through in binary scales of notation. Hence conversions from decimal to binary, and from binary to decimal, may be done on the machine. During code-checking, for example, it frequently happens that intermediate results are needed. Also, if there are first a few input data, or if one wants to change one or a few numerical constants in a program read into the machine, a conversion table might be useful.

Because of the large number of digits in the binary scale, a hexadecimal system, to base 16, is widely used instead, both systems being equivalent. In usual notation $A$, $B$, $C$, $D$, $E$, $F$ are respectively used for 10, 11, 12, 13, 14, 15.

Fröberg's pamphlet contains four tables. In Table I (pp. 5–9) for the integers 0(1)1024(16)4096 the corresponding hexadecimal numbers are given. Examples: for $941\sim 3AD$; for $2128\sim 850$; for $4016\sim FB0$. Table II (pp. 10–17)

gives the hexadecimal equivalents of the decimal fractions $0.{}^n01(0.{}^n01)0.{}^{n-1}1$, $n=0(2)14$. Examples:

$$.01 \sim .028F\,5C28\,F5C2\,9; \quad 0.{}^{10}51 \sim .0000\,0000\,3813\,4.$$

Table III (p. 18) is of frequently used constants in hexadecimal form. $e \sim 2.B7E1\,5762\,8AED\,1$ is worked out by means of the given tables. Examples:

$$\log 2 \sim .4D10\,4D42\,7DE8; \quad \ln 2 \sim B172\,17F7\,D1CF.$$

In Table IV (pp. 19–20) are various hexadecimal fractions (decimal points omitted) converted to decimal forms.

In this connection it may be of interest to recall the elaborate octal scale table of A. van Wijngaarden [Computation Dept., Math. Centrum, Amsterdam, Rep. **R130** (1951); these Rev. **13**, 386]. *R. C. Archibald.*

★**Tables for the analysis of beta spectra.** National Bureau of Standards Applied Mathematics Series, no. 13. U. S. Government Printing Office, Washington, D. C., 1952. iii+61 pp. $0.35.

These tables were designed to assist in the theoretical analysis of experimental data on beta-ray spectra.

The principal tables give values of the "Fermi function"

$$f(Z,\eta)=\eta^{2+2S}e^{\pm\pi\delta}|\Gamma(1+S+i\delta)|^2$$

for $Z=1(1)100$, $\eta=0(.05)1(.1)7$. The following notations are used: $\epsilon=(1+\eta^2)^{1/2}$, $T=510.91(\epsilon-1)$, $\beta=\eta/\epsilon$, $\gamma=Z/137$, $S=(1-\gamma^2)^{1/2}-1$, $\delta=\gamma/\beta$, $B\rho=1704.3\eta$. Each $Z$ has a column to itself (going over two adjacent pages). The column heading gives values of $Z$, $\gamma$, $S$, and $\varphi(Z)$, and each row (for a given $\eta$, $T$, $\epsilon$, $B\rho$, all of which are shown) gives the values of $f$ indicated as $\beta^-$ and $\beta^+$ for the upper and lower sign in the exponential respectively. At the foot of each column, the constants $A$, $B$, $C$ are listed, and computation from the approximate formula

$$f(Z,\eta)\approx\eta^{2+2S}\left(A+\frac{B}{\eta}+\frac{C}{\eta^2}\right)$$

is recommended for $\eta>7$. On p. 10, $\varphi(Z)$ is defined as

$$(4\pi mcR/h)^{2S}[\Gamma(3)/\Gamma(3+2S)]^2(1+S/2),$$

and this definition is used on pp. 11–21. The values given as $\varphi(Z)$ in the tables on pp. 22–61 are really those of

$$\left(\frac{2+2S}{3+2S}\right)^2\varphi(Z).$$

This is pointed out in a separate mimeographed sheet which should accompany each copy: this sheet also contains a correction to $\varphi(100)$, and lists the values of $\varphi(Z)$ for $Z=1(1)100$.

Auxiliary tables are included. Table 1 gives values of $\eta$, $\epsilon$, $\beta$, $T$ for $B\rho=0(100)20000$; Table 2, values of $\epsilon$, $\eta$, $\beta$, $B\rho$ for $T=0(10)60(20)200(50)1000(100)5000(200)10000$; and tables 3 to 6 are other, shorter, auxiliary tables. A discussion of beta spectra by U. Fano explains the use of the tables and their computation. *A. Erdélyi* (Pasadena, Calif.).

**Falkner, V. M. Tables of Multhopp and other functions for use in lifting-line and lifting-plane theory.** Appendix by E. J. Watson. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2593** (11,234), 52 pp. (1952).

The report gives the derivation and computed tables of two classes of functions suitable for the solution of problems of spanwise aerodynamic loading of wings either by lifting-line or lifting-plane theory. The functions are based on lifting-line theory, but, by a consideration of the connection between lifting-line and lifting-plane theory through the application of Munk's stagger theorem to the calculation of induced drag, it is deduced that the functions must be equally suitable for lifting-plane theory. The first range of functions, called Multhopp or $M$ functions, is associated with discontinuities of induced downwash, while the second, called $P$ functions because of the polygonal representation of induced downwash, is connected with discontinuities in rate of change of induced downwash. Examples are given of the combination of functions to produce given curves of induced downwash, and evidence of the close relation between the results for a continuous and stepped downwash curve suggests that the functions tabulated will be sufficient to cover almost any problem in wing loading. (From the author's summary.) *E. Reissner* (Cambridge, Mass.).

**Purchvanidze, A. V. An approximate formula for computation of the ratio of the areas of an elliptic sector and a triangle.** Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **5**, 212–215 (1952). (Russian)

**Uhler, Horace S. Many-figure approximations for $\sqrt[3]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{4}$, and $\sqrt[3]{9}$ with $\chi^2$ data.** Scripta Math. **18**, 173–176 (1952).

**Storchi, Edoardo. Espressione generale di $\pi$ in somme di arcotangenti.** Ann. Mat. Pura Appl. (4) **33**, 381–396 (1952).

The author discusses the general theory of the resolution of $\pi/4=\operatorname{arccot} 1$ into the sum of quantities $\operatorname{arccot} x$, denoted by $[x]$ in Lehmer's notation [Amer. Math. Monthly **45**, 657–664 (1938)]. He is interested primarily but not exclusively in integral values of $x$. Among his general relations are expressions for the solution $x$ of $[1]=n[a]+[b]+[x]$, and the formula

$$[1]=[p]+[q]+[r]+[(S_3+S_2-S_1-1)/(S_3-S_2-S_1+1)]$$

where $S_1$, $S_2$, $S_3$ are defined by

$$t^3-S_1t^2+S_2t-S_3=(t-p)(t-q)(t-r).$$

Among the many numerical examples are

$$[1]=3[4]+[20]+[1986]+[3942211]$$

which resembles the formula $[1]=3[4]+[20]+[1985]$ used in computing $\pi$ by D. F. Ferguson [Math. Gaz. **30**, 89–90 (1946)]. The author might well have given reference to Lehmer's paper above, to other pertinent papers [Frame, Amer. Math. Monthly **42**, 499–501 (1935); Wrench, ibid. **45**, 108–109 (1938); Ballantine ibid. **46**, 499–500 (1939)], and to Reitwiesner's account [Math. Tables and Other Aids to Computation **4**, 11–15 (1950); these Rev. **12**, 286] of the 2035D computation of $\pi$ by the Eniac. *J. S. Frame* (East Lansing, Mich.).

**Niven, Ivan. On the error term in interpolation formulas.** Quart. Appl. Math. **10**, 397–398 (1953).

**Lah, Ivo. Noch einige praktische Interpolationsformeln des Zinsfussproblemes von hoher Präzision.** Mitt. Verein. Schweiz. Versich.-Math. **52**, 161–172 (1952).

The author continues his earlier investigations of interpolation formulae for annuities [same Mitt. **51**, 91–100 (1951); these Rev. **12**, 752]. He introduces Poukka's ratios of higher order into the Taylor series for the annuities and combines then various simple interpolation formulae. By these operations formulae of greater accuracy are obtained. *E. Lukacs* (Washington, D. C.).

**Faddeeva, V. N. Computational methods of linear algebra. Chapter 1. Basic material from linear algebra.** Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., NBS Rep. **1644**. vi+93 pp. (1952).

Translated from Faddeeva's book [Gostehizdat, Moscow-Leningrad, 1950; these Rev. 13, 872].

**Lopshitŝ, A. M. A numerical method for determining the characteristic values and characteristic planes of a linear operator.** Four articles on numerical matrix methods, pp. 1–43. Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., NBS Rep. **2007** (1952).

Translated from Trudy Sem. Vektor. Tenzor. Analizu **7**, 233–259 (1949); these Rev. 13, 991.

**Gavurin, M. K. The application of polynomials of best approximation to the improvement of the convergence of iterative processes.** Four articles on numerical matrix methods, pp. 44–50. Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., NBS Rep. **2007** (1952).

Translated from Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 156–160 (1950); these Rev. 12, 209.

**Birman, M. SH. Some estimates for the method of steepest descent.** Four articles on numerical matrix methods, pp. 51–56. Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., NBS Rep. **2007** (1952).

Translated from Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 152–155 (1950); these Rev. 12, 32.

**Azbelev, N., and Vinograd, R. A process of successive approximations for finding characteristic values and characteristic vectors.** Four articles on numerical matrix methods, pp. 57–60. Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., NBS Rep. **2007** (1952).

Translated from Doklady Akad. Nauk SSSR (N.S.) **83**, 173–174 (1952); these Rev. 14, 126.

**van den Dungen, F. H. Principe de Rayleigh et regula falsi de Newton.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 695–704 (1952).

In order not to conceal the objective of the paper by an involved notation the author carries through his demonstration only for a specific case, that of the frequencies of a vibrating beam simply supported at the ends. As one method he obtains the characteristic equation and applies Newton's regula falsi to obtain approximations to a root. As a second method he applies Rayleigh's principle using an arbitrary deformation of the elastic system. The point of the paper is that starting from the same first guess $\lambda_0$ both methods give exactly the same improved value $\lambda_1$ after one step, and hence give the same sequence throughout. *W. E. Milne.*

**Collatz, Lothar. Zur numerischen Bestimmung periodischer Lösungen bei nichtlinearen Schwingungen.** Z. Angew. Math. Physik **3**, 193–205 (1952).

The approximate determination of periodic solutions in the case of nonlinear oscillations usually leads to the problem of solving a set of nonlinear algebraic equations to obtain either undetermined coefficients or functional values. In most cases this is a laborious process. However, the author gives examples where a trigonometric solution is assumed in which the coefficients can be found without excessive labor by relaxation. Also for equations of the type $L[x]=N[x^{(i)},t]$ where $L[x]$ is a linear differential expression with constant coefficients and $N$ is nonlinear he presents and illustrates methods for finding a set of approximate values of $x$ at equally spaced values of $t$. *W. E. Milne.*

**Salzer, Herbert E. On calculating the zeros of polynomials by the method of Lucas.** J. Research Nat. Bur. Standards **49**, 133–134 (1952).

When $f(x)$ is a polynomial of degree $n$ and $x_i$, $i=0, 1, \cdots, n$, are any $n+1$ points at which $f(x_i)\neq 0$, the zeros of $f(x)$ are known to be identical with the zeros of $\sum a_i/(x-x_i)$, where $a_i=f(x_i)/\prod'(x_i-x_j)$. Lucas proposed this principle for use in an electric analogue device for finding zeros. The present note evaluates this principle in digital computation for both real and complex zeros when the coefficients of $f(x)$ are given exactly (integral or rational) so that the zeros of $f(x)$ are identical with the zeros of $\sum A_i/(x-i)$, $A_i$ integral. The chief advantages are (1) the saving of labor in tabulating $\sum A_i/(x-i)$ instead of $f(x)$ in the neighborhood of the zero, especially for complex zeros, and (2) somewhat less work in the inverse interpolation for the zero. Three examples in locating a real root, and one example in locating a complex root were worked out in support of these findings. (From the author's summary.) *J. C. P. Miller.*

**Wenzl, Fritz. Zur numerischen Auflösung algebraischer Gleichungen.** S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 81–111 (1952).

A method is described for the solution of algebraic equations of fourth degree which depends on factorization into two second degree factors (cf. the method of B. Friedman [Comm. Pure Appl. Math. **2**, 195–208 (1949); these Rev. 11, 402]). Conditions of convergence are obtained. The method is extended to equations of higher degree.

*E. Frank* (Chicago, Ill.).

**Barrett, W. On the remainders of numerical formulae, with special reference to differentiation formulae.** J. London Math. Soc. **27**, 456–464 (1952).

Many numerical formulae have remainders of the form

$$R=\int_{-\infty}^{\infty} f^{(r+1)}(t)\varphi(t)dt$$

in which $f(t)$ is the function to which the formula is applied and $\varphi(t)$ is a function depending only on the particular formula. If $\varphi(t)$ does not change sign, the remainder term has a particularly simple form. The author establishes criteria which in many cases determine whether or not $\varphi(t)$ changes sign. With the aid of these he examines a number of quadrature and differentiation formulae, obtaining remainder terms which in general are already known for the case of quadrature formulas but lead to new and interesting results for successive derivatives of Bessel's and Stirling's interpolation formulas. *W. E. Milne* (Corvallis, Ore.).

**Barrett, W. On the remainder term in numerical integration formulae.** J. London Math. Soc. **27**, 465–470 (1952).

This paper continues the study of remainder terms begun in the preceding article, applying the investigation to Gregory's formula and to Weddle's rule, for both of which the function $\varphi(t)$ changes sign. Practical estimates of the remainder are found for both. *W. E. Milne.*

**Luke, Yudell L. Mechanical quadrature near a singularity.** Math. Tables and Other Aids to Computation 6, 215–219 (1952).

Evaluation approchée de $\int_0^b x^{-1/2} f(x)dx$ au moyen de $n+1$ ordonnées équidistantes: $0, h, \cdots, nh=b$ (formule exacte si $f(x)$ est un polynôme de degré $n$). Evaluation du reste par la méthode de Milne [Numerical calculus . . . , Princeton, 1949, pp. 108–116; ces Rev. 10, 483]. Tableau donnant les coefficients exacts pour $n=1, \cdots, 10$. Données permettant l'évaluation de l'erreur pour les mêmes valeurs de $n$. Discussion des publications antérieures sur la question, en particulier, E. L. Kaplan [J. Math. Physics 31, 1–28 (1952); ces Rev. 13, 782]. *J. Kuntzmann* (Grenoble).

**Mineur, Henri. Tentatives de calcul numérique des intégrales doubles.** Ann. Astrophysique 15, 54–70 (1952).

Details are furnished to the author's previous paper [C. R. Acad. Sci. Paris 233, 1166–1168 (1951); these Rev. 13, 588], in which he announced certain analogues in the case of double integrals over a square of Gauss' method of quadrature. Cubature formulas are obtained for 3, 4, 9, and 13 points, respectively. *P. W. Ketchum* (Urbana, Ill.).

**Brock, P., and Murray, F. J. The use of exponential sums in step by step integration. II.** Math. Tables and Other Aids to Computation 6, 138–150 (1952).

[For part I see same journal 6, 63–78 (1952); these Rev. 13, 873.] The error analysis for the exponential method is predicated on the convergence of the series expansion of an explicitly given function. The authors provide a table for computing the radius of convergence in cases of complex $\lambda$. Methods for accurate calculation of the coefficients in the open type formulas are studied in detail. The paper closes with some practical problems to which the method of exponential sums has been successfully applied.

*W. E. Milne* (Corvallis, Ore.).

**Löwdin, Per-Olov. On the numerical integration of ordinary differential equations of the first order.** Quart. Appl. Math. 10, 97–111 (1952).

It is a fact familiar to those interested in the numerical solution of differential equations that quadrature formulas using central differences are more accurate than corresponding backward difference formulas but require more extensive extrapolation. The aim of Löwdin's paper is to retain the good points and as far as possible avoid the bad points of both central and backward differences.

For the differential equation $dy/dx=F(x, y)$ the formula adopted is

$$(1)\quad h^{-1}y-\tfrac{1}{2}F(x, y)=(\mu\delta)^{-1}\{F-\tfrac{1}{2}\nabla F\} -\frac{1}{180}\mu\delta^3F+\frac{31}{15120}\mu\delta^5F-\frac{557}{907200}\mu\delta^7F+\cdots.$$

Here $\mu$ and $\delta$ are Sheppard's difference operators and the operator $(\mu\delta)^{-1}$ denotes summation. The good feature of this formula is that the part requiring extrapolation is quite small for any "respectable" differential equation. The author also gives formulas for performing this extrapolation. A possible bad feature is the necessity of solving the equation (1) for $y$ when the right-hand member is assumed known. It is recommended that this step (for non-linear equations) be performed by iteration. The explanation is detailed, and is illustrated by a numerical example. *W. E. Milne.*

**Šura-Bura, M. R. Estimates of errors of numerical integration of ordinary differential equations.** Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 575–588 (1952). (Russian)

The author obtains bounds for the error after $n$ steps in the step-by-step numerical integration of $y'=f(x, y)$, first by the use of an extrapolatory quadrature formula, and second by an interpolatory quadrature formula. The results apply, as special cases, to Adams' method and to Milne's method. The derivation employs the roots of a characteristic equation associated with a certain difference equation. The method is extended to the case of systems of first order equations. *W. E. Milne* (Corvallis, Ore.).

**Zurmühl, Rudolf. Runge-Kutta-Verfahren unter Verwendung höherer Ableitungen.** Z. Angew. Math. Mech. 32, 153–154 (1952).

In recent times several methods of numerical solution of differential equations have appeared which endeavor to increase the accuracy without increasing the number of steps by making use of additional derivatives obtained by differentiating the differential equation itself. In the present paper the author extends this idea to the Runge-Kutta process, obtaining appropriate formulas and setting up a suitable computational routine. *W. E. Milne.*

**Blanch, Gertrude. On the numerical solution of equations involving differential operators with constant coefficients.** Math. Tables and Other Aids to Computation 6, 219–223 (1952).

The equations to which the method of the paper applies are of the form

$$(1)\qquad Ly=F(y, x)$$

where $L$ denotes a linear differential operator. The procedure is to set up the integral equation for (1) by the method of variation of parameters, just as though the right-hand member were independent of $y$. The integral is then evaluated by some suitable quadrature formula using the equally spaced ordinates $y_n, y_{n-1}, \cdots, y_{n-k}$. Of these $y_{n-1}, \cdots, y_{n-k}$ are known from previous steps, and it turns out (this is the crux of the paper) that the coefficient of the term $F(y_n, x_n)$ vanishes, so that the integral can be evaluated without previous knowledge of $y_n$. Thus a value of $y_n$ is obtained by simple quadratures and the computation proceeds step by step. *W. E. Milne* (Corvallis, Ore.).

**Flügge, S. Zur numerischen und graphischen Integration von Schwingungsgleichungen.** Z. Physik 133, 449–450 (1952).

The usual method of step-by-step numerical integration is not well adapted to the solution of $y''+f^2(x)y=0$ in cases where $f(x)$ is large. For such cases the author proposes the recursion formulas

$$y_{n+1}=y_n\cos(f_n\Delta x_n)+\frac{y'_n}{f_n}\sin(f_n\Delta x_n)$$

$$\frac{y'_{n+1}}{f_n}=-y_n\sin(f_n\Delta x_n)+\frac{y'_n}{f_n}\cos(f_n\Delta x_n).$$

*W. E. Milne* (Corvallis, Ore.).

**Vorob'ev, Yu. V. A method of numerical integration of a class of equations of mathematical physics and its application to problems of electron optics.** Akad. Nauk SSSR. Žurnal Tehn. Fiz. 22, 1166–1173 (1952). (Russian)

This paper treats the equation

$$(1)\qquad y''+Q(x)y=0$$

where $Q(x)=P(x)/x^2$, $P(0)\neq 0$, so that the equation has a regular singular point at the origin. The general idea of the solution is to assume $y=x^\rho g(x)/P(x)$, where $\rho$ is a root of the indicial equation $\rho(\rho-1)+P_0=0$. Then $g(x)$ is expanded in terms of factorials with coefficients which are determined to satisfy (1). This provides an approximate solution in the vicinity of the singular point. The solution is then continued by ordinary step-by-step integration. The author applies the method to a differential equation arising in electron optics.
*W. E. Milne* (Corvallis, Ore.).

**Wiskott, D. Eine graphische Methode zur Integration der Schrödinger-Gleichung.** Z. Physik 133, 443–448 (1952).

The Schrödinger equation in one dimension

$$y''+2mh^{-2}(E-V(x))y=0$$

is solved by a graphic construction in the complex $y$-plane. The construction is based on the fact that the solution of $y''+k^2y=0$ ($k$ constant) runs around an ellipse in the $y$-plane as $x$ varies through real values. The general case is treated by hitching together short arcs of successive ellipses.
*W. E. Milne* (Corvallis, Ore.).

**Kohn, W. Variational methods for periodic lattices.** Physical Rev. (2) 87, 472–481 (1952).

Um Lösungen der zugehörigen Schrödingergleichung zu finden und diskutieren zu können, wird heir [im Gegensatz zu Slepian, thesis, Harvard University, 1949, der ein Variationsproblem mit Zwangsbedingungen formulierte] ein Variationsproblem mit Randfunktion und freien Randbedingungen formuliert. Im eindimensionalem Fall wird dieses Variationsproblem insbesondere auch benützt, um die von Kramers [Physica 2, 483–490 (1935)] gewonnenen Resultate über die typische Form der $E_n(k)$-Kurven ($E$ Energiekonstante, $k$ Wellenzahl) neuerdings abzuleiten. Auch die numerische Behandlung nach Rayleigh-Ritz wird besprochen und durch ein Beispiel illustriert. Im dreidimensionalen Fall wird die Beziehung zu den Näherungsmethoden von Wigner-Seitz-Bardeen [Wigner und Seitz, Physical Rev. (2) 43, 804–810 (1933); Wigner, ibid. 46, 1002–1011 (1934); Seitz, ibid. 47, 400–412 (1935); Bardeen, J. Chem. Phys. 6, 367–371 (1938)], Kuhn-Van Vleck [Kuhn und Van Vleck, Physical Rev. 79, 382–388 (1950); Silverman und Kohn, ibid. 80, 912–913 (1950); Silverman, ibid. 85, 227–230 (1952)], und Slater [ibid. 45, 794–801 (1934)] erläutert und insbesondere wird die Überlegenheit der Variationsmethode gegenüber der Methode von Slater dargelegt. *P. Funk* (Wien).

**Agodi, Attilio. Un metodo per il calcolo approssimato degli autovalori relativi ad un particolare sistema di Sturm-Liouville.** Boll. Accad. Gioenia Sci. Nat. Catania (4) no. 8, 487–497 (1951).

Der Verfasser setzt die Lösung der Schrödingergleichung (die dem Tröpfchenmodell entspricht) in Kugelkoordinaten wie üblich in Form eines Produktes einer nur vom Radius $r$ abhängigen Funktion $R(r)$ und einer Kugelflächenfunktion an. Für die für $R(r)$ sich ergebende Differentialgleichung wird die Methode von Fubini [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 26, 253–259 (1937)] herangezogen. *P. Funk* (Wien).

***King, Gilbert W. Stochastic methods in quantum mechanics.** Proceedings, Seminar on Scientific Computation, November, 1949, pp. 42–48. International Business Machines Corp., New York, N. Y., 1950.

The time-independent Schrödinger's equation can be looked upon as arising from a real diffusion problem rather than from the time-dependent Schrödinger equation with an imaginary coefficient. With this reformulation the problem of finding eigenvalues and eigenfunctions can be reduced to statistical calculations on suitably chosen random ~~values~~ walks. The method described in the present paper is closely related to the method of Donsker and the reviewer [J. Research Nat. Bur. Standards 44, 551–557 (1950); these Rev. 13, 590]. Connection between the present method and that based on difference equations is discussed. Numerical results for the harmonic oscillator and a particle in a box are given.
*M. Kac* (Ithaca, N. Y.).

**Laudet, Michel. Calcul numérique du potentiel et du flux d'un disque uniformément chargé.** J. Phys. Radium (8) 13, 549–553 (1952).

**Bohan, N. A. An attempt at numerical integration on punched-card machines of the equations of motion of minor planets for a given system of initial conditions.** Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. 5, 203–211 (1952). (Russian)

**Samoĭlova-Yahontova, N. S., and Makover, S. G. Computation of special perturbations of minor planets on punched-card machines.** Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. 5, 125–183 (1952). (Russian)

**Silva, Giovanni. Sulla determinazione pratica dei coefficienti di un polinomio di funzioni sferiche.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 643–649 (1952).

The aim of this note is to extend the idea of harmonic analysis to the case of a function $f$ defined on the surface of a sphere and for which an approximate representation is sought in terms of surface spherical harmonics

$$f=Y_0+Y_1+\cdots+Y_n. \tag{1}$$

The explicit expression in terms of latitude and longitude for the right-hand member of (1) contains $N$ terms with $N$ disposable constants, where $N=(n+1)^2$. The gist of the paper consists in trying to devise a scheme for selecting $N$ points on the unit sphere at which to apply least squares for the determination of the constants, the desired objective being to make the off-diagonal elements vanish in the matrix of normal equations. *W. E. Milne* (Corvallis, Ore.).

## ASTRONOMY

**Bragard, L. Sur le problème fondamental de la géodésie dynamique. I.** Bull. Soc. Roy. Sci. Liège 21, 247–258 (1952).

The deviation along a radius ($\Delta r$) of the cogeoid ("compensated geoid") from a figure of reference and the differences in gravity are expressed in terms of spherical harmonics. This enables the author to derive an integral equation which must be satisfied by $\Delta r$ in the case where the gravity centers of the cogeoid and of the figure of reference coincide. Another Fredholm equation for non-coincidence of these points and the solutions are also given.
*W. Jardetzky* (New York, N. Y.).

**Bragard, Lucien. Expressions des composantes de la déviation absolue de la verticale sur le cogéoïde.** C. R. Acad. Sci. Paris **234**, 2157–2159 (1952).

From the solution of the problem discussed in the preceding note the N–S and E–W components of the deviation of a vertical are derived for the case of non-coincidence of the gravity centers of the cogeoid and of the figure of reference, respectively. *W. Jardetzky* (New York, N. Y.).

**Mineo, Corradino. Teoria idrostatica delle configurazioni d'equilibrio dei pianeti fluidi rotanti e teoria di Stokes nel caso particolare della Terra.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 635–642 (1952).

The purpose of the paper is to stress the essential points of two problems and their solutions, namely, that concerning the figures of equilibrium of rotating fluid masses and the Stokes problem. These considerations of the author are to show that there were no confusions in earlier investigations despite the remarks of G. B. Pacella [same Rend. (8) **11**, 69–73 (1951)]. *W. S. Jardetzky* (New York, N. Y.).

**Stumpff, K. Hauptgleichung und Entwicklungssätze in punktmechanischen Problemen, insbesondere in der Zweikörperbewegung.** Astr. Nachr. **280**, 97–112 (1952).

Questo lavoro perfeziona due altri lavori dello stesso autore [Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1947**, no. 1 (1949); Astr. Nachr. **275**, 108–128 (1947); questi Rev. 11, 545; **10**, 745]. Qui viene mostrato il legame del metodo usato nelle precedenti memorie ai principi generali della dinamica, ciò che permette senz'altro, sulla base di questi, di estendere quei risultati a leggi di forza qualisivoglia-no. L'autore indica dei criteri di rappresentazione in forma chiusa degli integrali mediante variabili definite da quella che egli chiama "Hauptgleichung". I risultati vengono illustrati nel caso classico del problema dei due corpi sia quando la legge di attrazione è quella di Newton, sia per altri tipi di attrazioni. *G. Lampariello* (Messina).

**Goldstein, Allen A., and Fröberg, Carl-Erik. A collision path from the earth to the moon in the restricted problem of three bodies.** Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **22**, no. 14, 3 pp. (1952).

The numerical integrations reported in this paper were made on the SEAC, the Standards Eastern Automatic Computer, at the National Bureau of Standards, Washington, D. C. The problem considered is that of the motion of a body with infinitesimal mass, attracted by the earth and the moon. The center of mass of the earth-moon system was used as the origin of a coordinate system rotating with the sidereal mean motion of the moon. The two finite bodies were assumed to revolve in circular orbits. The equations of motion are therefore those of the restricted problem. Only trajectories in the $(x, y)$-plane are considered. A point on the surface of the earth and an initial velocity about 1.5 times the velocity of escape were chosen. The paper gives details on the integration procedure, on the units chosen, and on the numerical values of constants. The collision orbit could be found in three trials. For this trajectory, a table of results gives coordinates and velocity components as well as the computed values of the Jacobian integral for the starting point, the end point, and nine intermediary points. *D. Brouwer* (New Haven, Conn.).

**Thüring, B. Die Librationsbahnen der Trojaner als nichtgeschlossene Bahnkurven.** Astr. Nachr. **280**, 226–232 (1952).

By use of the Jacobi integral for the restricted three-body problem, it is shown that the Trojan asteroids cannot execute periodic oscillations about the libration points and must in fact follow certain spirals. It is shown how the Jacobi constant can be introduced as one orbit element in the generalized restricted problem for which the finite masses move in ellipses rather than circles. As the element varies, one finds a varying restriction on the part of phase space in which the infinitesimal mass can move. *W. Kaplan* (Ann Arbor, Mich.).

**Chazy, Jean. Sur la valeur d'un déterminant fonctionnel de la mécanique céleste.** Rend. Circ. Mat. Palermo (2) **1**, 28–34 (1952).

Let $y_i$ $(i=1, \cdots, 6)$ be the 6 variables specifying rectangular coordinates and velocities of a particle; let $C_i$ $(i=1, \cdots, 6)$ be the 6 elliptic elements and let $J$ be the Jacobian of the $C_i$ with respect to the $y_i$. The problem is to compute $|J|$. Let the equations of motion of a planet be written in the standard form (1) $\dot{C}_i=\sum_k p_{ik}(\partial R/\partial C_k)$, in terms of the disturbing function $R$. It is shown that $\det(p_{ik})=J^{-2}$. A very short derivation of (1) in explicit form is given, with the aid of canonical elements obtained by the Jacobi method. The quantities $p_{ik}$ can be read off at once and a simple calculation yields $|J|$. *W. Kaplan.*

**Elenevskaya, N. B. Expansion of a perturbation function in a Fourier series with respect to the inclination. I. Expansion of the perturbation function in the spatial circular restricted problem of three points by means of Newcomb's method.** Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **5**, 69–96 (1952). (Russian)

**Skabickiĭ, I. N. Application of the kinetic equation to star systems.** Leningrad. Gos. Univ. Učenye Zapiski **136**, Ser. Mat. Nauk **22**, Trudy Astron. Observ. **15**, 10–32 (1950). (Russian)

The author applies Liouville's equation, modified to include collisions according to the theory of Ambarzumian [same Zapiski **22**, Ser. Mat. Nauk **4**, 19–22 (1938)], to systems of spherical symmetry in both space and impulse variables. The effect of collisions on a steady state distribution is studied and worked out to a first order approximation in the time variable with the following results: (1) the space distribution does not change with time; (2) the velocity distribution tends towards a Maxwell-Boltzmann distribution; (3) the time of relaxation in globular clusters is of the order of $10^8-10^{10}$ years. *R. G. Langebartel.*

**Ogorodnikov, K. F. On the dynamics of the local system.** Leningrad. Gos. Univ. Učenye Zapiski **136**, Ser. Mat. Nauk **22**, Trudy Astron. Observ. **15**, 3–9 (1950). (Russian)

The author finds that a direct application of the galactic differential rotation theory involving the Oort constants leads (in the case of Keplerian motion) to an elliptic orbit with high eccentricity ($e=.97$) for the motion of the centroid of the local system about the galactic center. He regards as more attractive the alternative that the local system is rotating about its own axis as well as revolving about the galactic center, in which case, according to existing data the orbit of the centroid would be approximately circular if

the angular velocity of the local system about its axis is 3.3 times the angular velocity of the galactic rotation at the distance of 8000 parsecs from its center.

*R. G. Langebartel* (Urbana, Ill.).

**Kushwaha, R. S. Adiabatic pulsations of a particular model of the variable star.** Proc. Nat. Inst. Sci. India **18**, 461–466 (1952).

The author finds that for the stellar model whose density is given by $\rho_c(1-r/R)$ where $\rho_c$ is density at the center, $r$ the equilibrium value of the distance from the center, and $R$ the star's radius, the radial adiabatic oscillations are stable. The treatment rests on numerical integration.

*R. G. Langebartel* (Urbana, Ill.).

**Raychaudhuri, Amalkumar. Condensations in expanding cosmologic models.** Physical Rev. (2) **86**, 90–92 (1952).

By restricting himself to perturbations in which the cosmological line-element retains the special form

$$ds^2=-e^{\mu}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2)+e^{\nu}dt^2,$$

the author obtains a sufficient degree of stability to explain why very high density concentrations are not observed in our universe. *A. Schild* (Pittsburgh, Pa.).

## RELATIVITY

**Hogarth, J. E., and McCrea, W. H. The relativistically rigid rod.** Proc. Cambridge Philos. Soc. **48**, 616–624 (1952).

The object of this paper is to examine a new definition of a rigid rod in the special theory of relativity. The term rod is applied to a uniform one-dimensional body, all motions being restricted to this one dimension, so that only longitudinal stresses are concerned. Following a previous paper by McCrea [Sci. Proc. Roy. Dublin Soc. (N.S.) **26**, 27–36 (1952)], a rigid rod is defined (a) as one along which disturbances are transmitted with speed $c$, but in the present paper this definition is expanded. Condition (a) is to apply for all possible amplitudes of the accompanying strain and for all possible states of stress existing before the application of the disturbing forces, and it is supplemented by condition (b) that any change of configuration of a rigid rod is reversible and takes place without dissipation of energy. It is shown that this new definition is equivalent to an equation of elasticity

$$T=\tfrac{1}{2}m_0c^2[1-(L_0/L)^2],$$

where $L_0$ is the length and $m_0$ the mass per unit length of the unstressed rod, and $L$ is its length when at rest under tension $T$. Consequently, this equation provides both a necessary and a sufficient condition for rigidity in the sense defined. *G. J. Whitrow* (London).

**Hély, Jean. Sur la dynamique du point matériel doué de spin.** Revue Sci. **90**, 135–136 (1952).

A note on the classical dynamics of a particle with spin in the special theory of relativity. *N. Rosen* (Haifa).

**De Sloovere, H. Sur la stabilité au sens de Th. De Donder, des lois de la relativité restreinte.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 861–862 (1952).

**de Donder, Théophile. Le rôle des liaisons de solidité en relativité générale.** C. R. Acad. Sci. Paris **235**, 347–348 (1952).

The author gives in outline form a report on a method for deriving the field equations for the metric of a space-time subject to certain auxiliary conditions. *A. H. Taub.*

**Hlavatý, Václav. The elementary basic principles of the unified theory of relativity. A.** J. Rational Mech. Anal. **1**, 539–562 (1952).

This is the first of three papers in which the author proposes to examine, mainly from a geometrical point of view, the principles underlying the unified theory of relativity recently proposed by Einstein. The present paper discusses the consequences of introducing into the space-time $X_4$ an asymmetric fundamental tensor $g_{\lambda\mu}$. In the ideal 3-dimensional projective space associated with any given point of $X_4$, the symmetric and skew-symmetric parts of $g_{\lambda\mu}$ (the former assumed to be of signature $(---+)$), respectively define a quadric and a linear complex of lines. Two lines of each regulus of the quadric in general belong to the complex, and there is thus defined upon the quadric a skew quadrilateral of which two opposite vertices turn out to be real and the other two complex conjugates. The sides of the quadrilateral are therefore imaginary but the diagonals are real. The vertices represent vectors of $X_4$ and the diagonals two totally perpendicular bivectors. Various analytical results are obtained, and special cases are examined. A discussion of the physical meaning of the results is given in the last section and attention is drawn to the connection of this theory with that of spinors. [Note: The projective geometry associated with a symmetric tensor and a bivector in space-time was discussed by the reviewer in Proc. London Math. Soc. (2) **41**, 302–322 (1936).] *H. S. Ruse* (Leeds).

**Ingraham, Richard L. Conformal relativity.** Proc. Nat. Acad. Sci. U. S. A. **38**, 921–925 (1952).
**Ingraham, L. Conformal relativity.** Nuovo Cimento (9) **9**, 886–926 (1952).

[The first paper is a brief summary of the second paper.] The hyperspheres in the tangent spaces at each event of space-time may be mapped into points of a projective five-space with a real quadric. The points of the tangent spaces (hyperspheres of zero radius) lie on this quadric. Linear transformations of this quadric into itself correspond to conformal transformations of the tangent spaces. Conformal relativity as propounded by the author is the theory of the collection of these quadrics described by a symmetric conformal tensor $S_{\mu\rho}$ ($\mu,\rho=0,1,\cdots,5$ and of signature $(-++++-+)$) and a general connection $\Gamma^{\sigma}{}_{\mu\rho}$. The field equations are obtained by minimizing the integral of the scalar curvature formed from these two quantities. These results when written in terms of tensors of space-time involve a symmetric tensor $g_{\mu\nu}$, two vectors $\varphi_\mu$, $f_\mu$ and a scalar $\rho$. The author interprets $g_{\mu\nu}$ as the gravitational field, $f_\mu$ as the electromagnetic vector-potential and $\varphi_\mu$ and $\rho$ as two meson fields. The paper closes with some statements concerning a further generalization of the theory to include a spinor field which the author promises to expound in later papers. *A. H. Taub* (Urbana, Ill.).

**Kohler, Max. Die Formulierung der Erhaltungssätze der Energie und des Impulses in der allgemeinen Relativitätstheorie.** Z. Physik **131**, 571–602 (1952).

The author proposes a theory of gravitation based on the geometry of a 4-space with two metric tensors $g_{ij}$ and $\bar{g}_{ij}$,

the space being flat with respect to $\bar{g}_{ij}$. This means that his gravitational field is described by a symmetric second order tensor in a Minkowski space and that many of the basic physical ideas of general relativity are lost (although the author's field equations are formally the same). By means of the flat background metric it is possible to construct a gravitational energy-momentum tensor which is conserved. This is illustrated in the case of weak gravitational fields.
*A. Schild* (Pittsburgh, Pa.).

**Mikhail, F. I. The relativistic clock problem.** Proc. Cambridge Philos. Soc. **48**, 608–615 (1952).

The problem of the so-called "clock paradox", which arose in connection with the phenomenon of time dilatation in special relativity, is formulated by the author purely in terms of general relativity: if space-time is such that it admits two distinct geodesics through a single pair of events, it is required to compare the time-intervals between these events as assigned by two observers having these geodesics as world-lines. As the author points out, this problem differs from the classical treatment in which the observers are supposed to be at rest relative to each other initially and finally. Since orbits allowing this behaviour do not exist in general relativity for unchanged particles, clock synchronization is formulated by the universal interpretation of the element of interval $ds$. Owing to the complexity of the general problem, the author concentrates on a particular example in which the two observers are attached to two test-particles moving freely in the field of a gravitating mass (Schwarzschild metric). One of the particles makes complete revolutions in a circular orbit while the other moves radially outwards and inwards. The time-interval between successive encounters is shorter in the reckoning of the former than in that of the latter, the results being calculated to a third-order approximation. As the author remarks, there is no reason to expect that the two observers should assign equal times to the duration of their journeys. *G. J. Whitrow.*

**Scherrer, W. Wirkungsprinzipien zur Feldtheorie der Materie.** Helvetica Phys. Acta **25**, 501–504 (1952).

In two previous papers [same Acta **22**, 537–551 (1949); **23**, 547–555 (1950); these Rev. **11**, 467; **12**, 292] the author has modified the action principle leading to Einstein's equations by introducing the matter density into it, but with physically unsatisfactory consequences. He here merely enumerates other more complicated possible action principles involving the gravitational and electromagnetic fields and promises a discussion of their consequences in a future paper. *A. J. Coleman* (Toronto, Ont.).

**Scherrer, W. Metrisches Feld und vektorielles Materiefeld.** Comment. Math. Helv. **26**, 184–202 (1952).

In the paper reviewed above the author has considered the effect of introducing a scalar matter amplitude into the equations of general relativity theory. Here, he introduces a four-vector $\varphi^\rho$ instead of a scalar function and considers the variation problem

$$\delta\int(H+2\xi J+2\eta L+2\zeta M+2\epsilon F)\sqrt{-g}\,dx=0$$

where $\xi$, $\eta$, $\zeta$, $\epsilon$ are universal constants, $H=R\varphi_\rho\varphi^\rho$, $J=R_{\rho\sigma}\varphi^\rho\varphi^\sigma$, $L=\varphi^\sigma{}_{/\rho}\varphi^\rho{}_{/\sigma}$, $M=(\varphi^\rho{}_{/\rho})^2$, $F=\frac{1}{2}F_{\rho\sigma}F^{\sigma\rho}$, $R_{\rho\sigma}$ is the Ricci tensor, $R$ the curvature invariant, $F_{\rho\sigma}=\varphi_{\rho/\sigma}-\varphi_{\sigma/\rho}$, and the solidus indicates covariant differentiation. This seemingly rather complicated problem possesses a family of static spherically symmetric solutions. Imposing the restriction that the integral throughout space of the components of the energy-momentum tensor be finite, isolates two solutions of the family and these, in spherical coordinates, have the property that $\varphi^4$ is asymptotic respectively to $e/r$ and $-e/r$. *A. J. Coleman* (Toronto, Ont.).

**Galli, Mario. Le deformazioni relativistiche di un cilindro rotante. II.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 569–574 (1952).

Continuation of previous discussion [same Rend. (8) **12**, 86–92 (1952); these Rev. **14**, 98] without any significant addition. *J. L. Synge* (Dublin).

**Takeno, Hyôitirô. On the spherically symmetric space-times in general relativity.** Progress Theoret. Physics **8**, 317–326 (1952).

The author seeks ways of determining whether a $ds^2$ given in general space-time coordinates $(x^i)$ is reducible to the form

$$ds^2=-A(r,t)dr^2-B(r,t)(d\theta^2+\sin^2\theta d\phi^2)+C(r,t)dt^2,$$

and for that purpose uses the "characteristic system" introduced by him in an earlier paper [J. Math. Soc. Japan **3**, 317–329 (1951); these Rev. **13**, 985]. As an illustration, he applies his methods to the space-times of Weyl and Levi-Cività, Gödel and Silberstein. *H. S. Ruse* (Leeds).

**Takeno, Hyôitirô. On relativistic theory of rotating disk.** Progress Theoret. Physics **7**, 367–376 (1952).

The problem of the rotating disk consists in finding appropriate world lines for the particles of the disk in a space-time which is either curved or flat. In this paper flat space-time is used and, by assuming a group property for the composition of angular velocities, the author is led to the formulae

$$\rho'=\rho,\quad z'=z,\quad \phi'=[\phi-(ct/\rho)\tanh\lambda]\cosh\lambda,$$
$$t'=[t-(\rho\phi/c)\tanh\lambda]\cosh\lambda,$$

where $\lambda=\rho\omega/c$, $\omega$ being the constant angular velocity of the disk; here $\rho$, $z$, $\phi$, $t$ are cylindrical coordinates used by a fixed Galileian observer, the $z$-axis being the axis of rotation, and $\rho'$, $z'$, $\phi'$ are constants for any individual particle, so that the first three of these formulae define a congruence of world lines. The transformation from $(\rho\phi, ct)$ to $(\rho\phi', ct')$ is in fact a Lorentz transformation. The velocity of a particle is $v=\rho d\phi/dt=c\tanh\lambda$. The geometry of the disk is examined, the spatial distance between two adjacent particles being the Minkowskian separation of their world lines, measured orthogonally to them. The element of proper time for a clock on the disk is $dt\cdot\operatorname{sech}\lambda$. The results are compared with those of C. W. Berenda [Physical Rev. (2) **62**, 280–290 (1942); these Rev. **4**, 117] and E. L. Hill [ibid. **69**, 488–491 (1946); these Rev. **8**, 175]. [It may be remarked that this motion of the disk implies a drift of the particles, in the sense that two particles, initially close together, will become separated in the course of time.] *J. L. Synge* (Dublin).

**Gupta, Suraj N. Quantization of Einstein's gravitational field: General treatment.** Proc. Phys. Soc. Sect. A. **65**, 608–619 (1952).

In this paper the author carries out in a simple and elegant way the quantization of the exact non-linear field-equations of general relativity. The method is the following. The gravitational potentials are written in the form

$$g^{\mu\nu}=e^{\mu\nu}-\kappa\gamma^{\mu\nu}$$

where $\epsilon^{\mu\nu}$ is the constant metric tensor of a flat Minkowski space, $\kappa$ is the gravitational constant, and the $\gamma^{\mu\nu}$ are the new field variables. The Lagrangian when expressed in terms of the $g^{\mu\nu}$ does not involve $\kappa$ except for the constant multiplying factor $\kappa^{-2}$. However, when expressed in terms of the $\gamma^{\mu\nu}$, the Lagrangian becomes

$$L=-\frac{1}{4}\left[\frac{\partial\gamma_{\mu\nu}}{\partial x_\lambda}\frac{\partial\gamma_{\mu\nu}}{\partial x_\lambda}-\frac{1}{2}\frac{\partial\gamma_{\mu\mu}}{\partial x_\lambda}\frac{\partial\gamma_{\nu\nu}}{\partial x_\lambda}\right]+\kappa L'.$$

Here $L'$ is an expression which can be written down in closed form. All terms in $L'$ are of third or higher order in the $\gamma^{\mu\nu}$, and involve either zero or positive powers of $\kappa$. The part $L_0$ of $L$ which is written explicitly above is just the Lagrangian of the linearized approximation to general relativity, which the author discussed in an earlier paper [same Proc. 65, 161–169 (1952); these Rev. 13, 804]. He showed that the linearized theory could be quantized by the same method as is used for the Maxwell theory; the quantized linearized theory describes the dynamics of an assembly of non-interacting particles with zero mass and spin 2. Now the complete theory can be quantized by making use of the interaction representation. The field operators $\gamma^{\mu\nu}$ are made to obey the field-equations and commutation relations of the linearized theory with Lagrangian $L_0$, and the wave-function then satisfies a Schrödinger equation in which the "Hamiltonian" is just the operator $\kappa L'$ expressed as a function of the $\gamma^{\mu\nu}$ so defined. The effect of $\kappa L'$ is to produce an apparent interaction between two quanta of the gravitational field, corresponding to the non-linear character of the original field-equations. The quantized theory so constructed is an exact one and not merely a weak-field approximation. It is however obviously well adapted to calculations in which a weak-field approximation is made by treating $\kappa L'$ as a small perturbation.

The theory so far described is a theory of a pure gravitational field. The author shows how to quantize by the same method the theory of the gravitational field in interaction with the Maxwell field and with a matter-field obeying the Dirac equation. In this case the main part of the interaction appears in $L'$ as a term $-\frac{1}{2}\gamma^{\mu\nu}T_{\mu\nu}$, where $T_{\mu\nu}$ is the energy-momentum tensor of the Maxwell and matter fields. The author ends by calculating with second-order perturbation theory the gravitational self-energies of the photon and the electron. These calculations are incomplete and the correct results will be published in a subsequent paper.

The simplicity and practicality of this method of quantizing the gravitational field are surprising, because previous attempts [for example, P. G. Bergmann, Physical Rev. (2) 75, 680–685 (1949); these Rev. 10, 408] have encountered severe mathematical difficulties. The previous attempts were all based on the idea that the general covariance of the theory under all coordinate transformations must be explicitly preserved in the quantization. The present author abandons general covariance at the start by using the coordinate conditions of De Donder, $\partial g^{\mu\nu}/\partial x^\nu=0$, to fix the coordinate system. He ignores completely the interpretation of the $g^{\mu\nu}$ as the metric tensor of a Riemannian space. This raises an extremely important question of principle, namely, whether the idea of Riemannian space-time is appropriate only to a macroscopic picture of the universe, or whether we should still try to bring general covariance into the fundamental microscopic equations of quantum theory. There is no way of deciding this question except to try it both ways.

*F. J. Dyson* (Ithaca, N. Y.).

**Pirani, F. A. E., Schild, A., and Skinner, R. Quantization of Einstein's gravitational field equations. II.** Physical Rev. (2) 87, 452–454 (1952).

In an earlier paper [Physical Rev. (2) 79, 986–991 (1950); these Rev. 13, 306] the authors obtained an explicit expression for the Hamiltonian of the gravitational field and of the combined gravitational and electromagnetic fields in the general theory of relativity. In the present paper they obtain explicit expressions for the constraints, i.e., relations holding among the variables in the course of the motion because of the covariance of the field theories. In connection with the Hamiltonian and the constraints, one should also see the papers of Bergmann and his co-workers [e.g., Bergmann, Penfield, Schiller and Zatzkis, ibid. 80, 81–88 (1950); these Rev. 12, 292; Anderson and Bergmann, ibid. 83, 1018–1025 (1951); these Rev. 13, 411]. *N. Rosen* (Haifa).

**Vigier, Jean-Pierre. Sur la relation entre l'onde à singularité et l'onde statistique en théorie unitaire relativiste.** C. R. Acad. Sci. Paris 235, 869–871 (1952).

**Graef Fernández, Carlos. The Birkhoff gravitational field of a mass-point in arbitrary movement in Birkhoff's theory.** Revista Mexicana Fisica 1, 11–27 (1952). (Spanish)

On the assumption that the instantaneous gravitational field due to an arbitrarily moving point-mass $M$ in Birkhoff's theory [Bol. Soc. Mat. Mexicana 1, nos. 4–5, 1–23 (1944); these Rev. 6, 240] is independent of the acceleration of $M$, the author obtains for the gravitational tensor $h_{ij}$ a formula that may be written

$$h_{ij}=M(2v_iv_j-\Delta_{ij})/\{\sum\Delta_{mn}(x^m-\xi^m)v^n\}.$$

Here $(x^i)$ is the world-point at which $h_{ij}$ is calculated, $(\xi^i)$ is the world-point of $M$, $v^i$ the velocity 4-vector of $M$, and $\Delta_{mn}$ is the fundamental tensor diag $\{1,-1,-1,-1\}$ of galilean space-time. The $(\xi^i)$, $(v^i)$ have retarded values, gravitational effects being propagated with the velocity of light. This 4-dimensional formula is obtained from a previously established (3+1)-dimensional formula. The quantities appearing in it, and in particular the denominator, are discussed in terms of the 4-dimensional geometry of space-time. The d'Alembertian of $h_{ij}$ vanishes. *H. S. Ruse.*

**Alba Andrade, Fernando. Nonplanar orbits of a particle in the field of a rotating sphere in Birkhoff's theory.** Revista Mexicana Fisica 1, 38–43 (1952). (Spanish. English summary)

On the basis of Birkhoff's theory, the author considers the motion of a particle in the gravitational field of a rotating sphere. As a first approximation the particle is assumed to move in a circular orbit, and the motion of the plane of the orbit is then discussed. *H. S. Ruse* (Leeds).

**Maravall Casesnoves, Darío. Non-Euclidean metric of space-time in the interior of a mass of barotropic fluid with spherical symmetry.** Revista Mat. Hisp.-Amer. (4) 12, 138–150 (1952). (Spanish)

In the spherically symmetric statical line element of general relativity

$$ds^2=-e^\lambda dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+e^\nu dt^2$$

$\lambda$ and $\nu$ are functions of $r$, satisfying the usual field equations (with cosmological term omitted)

$$8\pi p=e^{-\lambda}(\nu'/r+1/r^2)-1/r^2,$$
$$8\pi\rho=e^{-\lambda}(\lambda'/r-1/r^2)+1/r^2,$$
$$p'=-\tfrac{1}{2}(p+\rho)\nu',$$

$p$, $\rho$ being pressure and density. (The last of these equations is misprinted in the paper.) From these the author obtains at once

$$\nu=2\int(p+\rho)^{-1}dp, \tag{*}$$

$$e^{-\lambda}=(8\pi pr^2+1)/(r\nu'+1), \tag{**}$$

$$1-8\pi\rho r^2=\frac{d}{dr}\left(\frac{8\pi pr^3+r}{1-2rp'/(p+\rho)}\right). \tag{***}$$

(The last of these equations is corrected here; it is misprinted in the paper.) Since there are only three equations for four unknowns, a fourth equation is supplied; this is either (a) a characteristic (barotropic) equation $\rho=f(p)$, or (b) a relation of the form $\rho=F(r)$. In either case (***) gives $p(r)$, (*) gives $\nu(r)$ and (**) gives $\lambda(r)$. Since these equations cannot be solved explicitly in general, the author linearises the field equations for $p/\rho$ small, obtaining

$$\nu'/r-\lambda/r^2=0,\quad \lambda'/r+\lambda/r^2=8\pi\rho,\quad p'/\rho=-\tfrac{1}{2}\nu'.$$

[The approximation is not clear to the reviewer, to whom it appears that the right-hand side of the first equation should be $8\pi p$.] These equations are easily integrated in case (b), and this is done for fluid spheres with $\rho=\text{const.}$, $\rho=k/r$, and $\rho=k/r^2$. The case of radiation is also treated ($p=\rho/3$). [The author gives no references; in particular, he does not mention that an exact solution for $\rho=\text{const.}$ was given by Schwarzschild [S.-B. Preuss. Akad. Wiss. **1916**, 424–434]; cf. R. C. Tolman, Relativity, thermodynamics and cosmology, Oxford, 1934, p. 246.] *J. L. Synge.*

# MECHANICS

**✶Levi-Cività, Tullio, e Amaldi, Ugo. Lezioni di meccanica razionale. Vol. I. Cinematica—principi e statica.** Nuova ed. N. Zanichelli, Bologna, 1950. xviii+816 pp. 5,000 Lire.

**✶Levi-Cività, Tullio, e Amaldi, Ugo. Lezioni di meccanica razionale. Vol. 2. Dinamica dei sistemi con un numero finito di gradi di libertà.** Nuova ed. N. Zanichelli, Bologna, 1951, 1952. Parte 1: ix+510 pp.; parte 2: ix+673 pp. 4,000+5,000 Lire.

The first edition of volume 1 was published in 1923, the second edition in 1930. The first edition of Parts 1 and 2 of volume 2 appeared in 1926 and 1927, respectively. In preparing this new edition the second author has restricted himself to minor changes aside from correction of known errors (". . . ho sentito il dovere di limitarmi a quei ritocchi, che potevo presumere sarebbero stati voluti o accettati da Lui [Levi-Cività] stesso.").

**✶Sommerfeld, Arnold. Mechanics. Lectures on theoretical physics, Vol. I.** Translated from the fourth German edition by Martin O. Stern. Academic Press, Inc., New York, N. Y., 1952. xiv+289 pp. (1 plate). $6.50.

This is the first of a set of six books, which together constitute a comprehensive treatise on classical theoretical physics. The work is based upon the lectures given by the author at the University of Munich, having been written after his retirement.

The present volume is devoted to the mechanics of particles and rigid bodies. The following list of chapter titles will suffice to indicate the nature of the subject matter. (1) Mechanics of a particle; (2) Mechanics of systems, principle of virtual work, and d'Alembert's principle; (3) Oscillation problems; (4) The rigid body; (5) Relative motion; (6) Integral variational principles of mechanics and Lagrange's equations for generalized coordinates; (7) Differential variational principles of mechanics; (8) The theory of Hamilton. At the end of the book there is a collection of 42 substantial problems for the reader, accompanied by 27 pages of hints as to the solutions.

The outstanding impression obtained from reading the book is that this is decidedly the mechanics of the physicist. The mathematical work is always handled efficiently and elegantly; but the mathematics involved does not go beyond the first year graduate level, and there are no problems serving only as occasions for the display of mathematical virtuosity. Also, although there are some interesting problems derived from mechanical engineering, very little attention is given to technological applications. On the other hand, there is a constant and usually highly successful effort to clarify the underlying physical laws, and their bearing upon particular mechanical phenomena. Mention should be made here of the extremely lucid discussions of the motion of particles with variable masses, of the motions of rigid bodies, and of the laws of relative motion. As the author's career would lead one to expect, a considerable amount of the illustrative material is drawn from problems of modern physics.

Although the book is devoted primarily to Newtonian mechanics, the discussion of the fundamental concepts is carried far enough to include good introductory expositions of the Lorentz transformation and of the elements of special relativistic mechanics.

In the opinion of the reviewer, the chapter on small oscillations is rather inadequate, for it consists almost entirely of discussions of a few examples, with virtually nothing in the way of general theory. As is also true in the cases of many other textbooks on mechanics, the illustrative material bearing on the Hamilton-Jacobi theory seems to be unduly scanty. The book is remarkably free from errors. On page 38 there is a table displaying the notations used by various authors in vector analysis. Here the indications of Gibbs' notation are not quite correct. The volume is excellently printed, and contains, as a frontispiece, a fine portrait of the author. *L. A. MacColl* (New York, N. Y.).

**✶Smart, E. Howard. Advanced dynamics. Vol. I. Dynamics of a particle. Vol. II. Dynamics of a solid body.** Macmillan and Co., London, 1951. Vol. I, xi+419 pp.; vol. II, xi+420 pp. $12.00.

Questa interessante opera di dinamica del punto materiale (vol. I) e di dinamica di un corpo rigido (vol. II) è il contenuto delle lezioni dell'autore all'Università di Londra. Essa è stata pubblicata a cura di F. G. W. Brown dopo che l'autore si è spento durante la sua composizione.

La materia del vol. I è distribuita in dieci capitoli nel modo sequente: Cap. I. Introduzione; Cap. II. Principi dinamici fondamentali; Cap. III. Moto rettilineo di un punto materiale; Cap. IV. Accelerazioni parallele ad assi rettangolari; Cap. V. Accelerazioni radiale e trasversale, forze centrali; Cap. VI. Moto libero sotto l'azione di una forza newtoniana; Cap. VII. Moto vincolato in due dimensioni; Cap. VIII. Moto in un mezzo resistente; Cap. IX. Moto di una catena in due dimensioni; Cap. X. Moto di un punto materiale in tre dimensioni.

La materia del vol. II è distribuita in tredici capitoli nel modo sequente: Cap. I. Momenti e prodotti d'inerzia; Cap. II. Cinematica piana; Cap. III. Il principio di d'Alembert e le equazioni del moto; Cap. IV. Moto in due dimensioni di un corpo rigido (forze finite); Cap. V. Moto in due dimensioni di un corpo rigido (forze impulsive); Cap. VI. Dimensioni e similitudine dinamica; Cap. VII. Moto in tre dimensioni; Cap. VIII. Moto in tre dimensioni (continuazione); Cap. IX. Equazioni di Lagrange; Cap. X. Moto in assenza di forze; Cap. XI. Moto di un giroscopio; Cap. XII. Equazioni di Hamilton; teoremi generali sugli impulsi, ecc.; Cap. XIII. Teoria delle piccole oscillazioni (continuazione del Cap. VIII).

Lo scopo che l'autore si è proposto di raggiungere e che ha felicemente raggiunto è stato di illustrare con una grande varietà di problemi ed esercizi bene scelti i principi fondamentali della dinamica di Galileo e Newton. L'autore si è costantemente sforzato di offrire allo studioso i mezzi necessari per applicare correttamente i principi. Una discussione sistematica di problemi di masse variabili accresce l'interesse dell'opera. In questo genere di problemi la seconda legge del moto di Newton dev'essere applicata con cautela.

Sebbene problemi di codesto tipo si trovino nelle classiche collezioni di esercizi della scuola inglese fin dal secolo scorso, la considerazione specifica di essi ai fini dell'enunciazione di una legge generale sembra dovuta a T. Levi-Civitá [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **7**, 329–333 (1928)] e A. Sommerfeld [Vorlesungen über theoretische Physik, Bd. 1, Akademie Verlagsgesellschaft, Leipzig, 1943; cfr. il libro recensito sopra].

Pur degne di considerazione sono le note storiche che opportunamente l'autore ha introdotte nel testo. Purtroppo nessun cenno trovasi della scoperta di Galileo del principio di relatività che, distrutto da Newton con l'introduzione del concetto metafisico di spazio assoluto, è stato la sorgente dei grandiosi sviluppi della fisica contemporanea per opera di Einstein. Ed è doveroso osservare che Galileo non si è limitato a riconoscere il principio, ma se ne è servito per stabilire la legge del moto dei gravi lanciati, dopo aver scoperto la legge della caduta libera dei gravi. Applicazione mirabile del principio di relatività che gli storici della meccanica, compreso il Mach (alla cui opera famosa attinge lo Smart), trascurano di far notare. *G. Lampariello.*

**Eremeev, N. V. On the theory of mechanisms of variable structure.** Moskov. Gos. Univ. Učenye Zapiski **154**, Mehanika **4**, 61–71 (1951). (Russian)

This vague paper seems to be concerned with the fact that Grübler's expression $(3n-2p)$ for the mobility of a plane linkage admits exceptions when the equations of the constraints happen to be dependent ("dwells" are a familiar example). The condition for this increased mobility is that the absolute and relative instant centers of any pair of links be collinear. This issue is at the basis of Artobolevskiĭ's concept of "class" which has already achieved textbook status in Russia. The purpose of the paper, therefore, is not clear. The author gives much emphasis to the well-known fact that increased mobility may be instantaneous (singular positions) or persistent. *A. W. Wundheiler.*

**Pismanik, K. M. On the momentary axis of worm gears.** Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **10**, no. 39, 5–15 (1951). (Russian)

A construction is given for the relative instantaneous axis of screw motion of a worm gear meshing with a circular gear, the angle of their axes being arbitrary. The construction is applied to a numerical design problem.

*A. W. Wundheiler* (Chicago, Ill.).

**Kolčin, N. I. Determination of radii of curvature of cams in cam mechanisms by the author's method of unround wheels.** Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **10**, no. 39, 16–21 (1951). (Russian)

Two cams with circular followers are considered. The center of one of them moves in a prescribed circular arc, that of the other moves in a straight line not going through the center of the cam. Graphical constructions are given for the radii of curvature of the cams for every position of the follower. The instantaneous equivalent four-bar linkage is used to justify the construction. The advantage of the method is that it uses velocities, and not accelerations.

*A. W. Wundheiler* (Chicago, Ill.).

**Dobronravov, V. V. On certain relations in the problem of motion of a rigid body about a fixed point in Euler's case.** Moskov. Gos. Univ. Učenye Zapiski **154**, Mehanika **4**, 55–59 (1951). (Russian)

The author of this unusual paper proposes to prove some "yet unnoticed" relations in the problem of the title. The first one (8) is equivalent to the theorem that the fixed angular-momentum vector moves, relative to the body, on the "invariable cone" [Routh, Advanced rigid dynamics, part II, 6th ed., Macmillan, London, 1905, p. 104] but the author's proof is more indirect than the classical one. The second relation (17) is obtained by renaming the fixed axes (one of them is directed along the angular momentum), using an even more indirect proof of the theorem mentioned, and substituting, for the directional cosines, the expressions in terms of Eulerian angles. Thus, a more impressive form of the first relation is obtained but it is not yet clear to the reviewer why (17) is not divided through by the factor $\sin\theta$ common to all its terms. No comment is made about the equivalence of (8) and (17). The third relation (19a) is very involved and links the Eulerian angles with their first derivatives. Since it contains no integration constants, the author concludes that (19a) is a "nonholonomic constraint". (The reviewer observes that the author's choice of one of the fixed axes reduces to zero two of the "reference" constants depending on the choice of the fixed frame. Thus (19a) is an involved form of one of two independent "constant-free" first integrals which exist in the Euler case without being very helpful.) There are other remarks in the paper.

*A. W. Wundheiler* (Chicago, Ill.).

**Rund, Hanno. Die Hamiltonsche Funktion bei allgemeinen dynamischen Systemen.** Arch. Math. **3**, 207–215 (1952).

By a general dynamical system is meant one in which the kinetic energy is not necessarily a quadratic form in the components of the velocity vector. The author adopts methods of Finsler geometry, as recently developed by himself, and arrives at a generalized form of the Hamilton-Jacobi theory. Notions such as the canonical equations, the Hamilton function, the Hamilton-Jacobi differential equation, etc., are introduced and geometrically interpreted.

*S. Chern* (Chicago, Ill.).

**Lichnerowicz, André, and Aufenkamp, Don. The general problem of the transformation of the equations of dynamics.** J. Rational Mech. Anal. 1, 499–520 (1952).

Let two dynamical systems be given by the following equations of motion:

$$\text{(D)}\qquad \frac{d^2x^\alpha}{dt^2}+\Gamma^\alpha{}_{\gamma\beta}\frac{dx^\gamma}{dt}\frac{dx^\beta}{dt}+a_\beta{}^\alpha\frac{dx^\beta}{dt}=Q^\alpha,$$

$$\text{(E)}\qquad \frac{d^2x^\alpha}{d\tau^2}+\Lambda^\alpha{}_{\gamma\beta}\frac{dx^\gamma}{d\tau}\frac{dx^\beta}{d\tau}+b_\beta{}^\alpha\frac{dx^\beta}{d\tau}=R^\alpha,$$

where $\Gamma^\alpha{}_{\gamma\beta}$ and $\Lambda^\alpha{}_{\gamma\beta}$ are the Christoffel symbols relative to the tensors $g_{\alpha\beta}$ and $h_{\alpha\beta}$, and $g_{\alpha\gamma}a_\beta{}^\gamma$ and $h_{\alpha\gamma}b_\beta{}^\gamma$ are both skew-symmetric. What are the conditions in order that the trajectories of the two systems be identical? In solving this problem the authors are led to distinguish two cases: a singular case, which appears to be identical with the case for which T. Y. Thomas solved the problem [J. Math. Physics 25, 191–208 (1946); these Rev. 8, 102], and a general case. It is shown that if in the general case (and not in the special case) any path of (E) is a path of (D), the converse is also true. *J. Haantjes* (Leiden).

**Trent, H. M. An alternative formulation of the laws of mechanics.** J. Appl. Mech. 19, 147–150 (1952).

For the calculation of the currents in an electric network, mesh-analysis may be preferable to nodal analysis. For the calculation of the internal forces in a mechanical system, the analogues of the mesh equations may be preferable to the usual particle equations which correspond to nodal analysis. The author presents a modified statement of the laws of mechanics from which such mesh equations may be directly formulated. *P. Franklin* (Cambridge, Mass.).

**Toupin, R. A. A variational principle for the mesh-type analysis of a mechanical system.** J. Appl. Mech. 19, 151–152 (1952).

The author derives the mesh equations of Trent [see the paper reviewed above] from a Lagrangian function, in which impulses play the role of the generalized coordinates of the classical theory. *P. Franklin.*

**Le Corbeiller, P., and Yeung, Ying-Wa. Duality in mechanics.** J. Acoust. Soc. Amer. 24, 643–648 (1952).

The duality is that relating the equations of a mechanical system in terms of forces (Newton-D'Alembert) to that in terms of velocities (Firestone-Trent). In this relation coordinate corresponds to momentum. The direct derivation of the second form, as well as the analogy with two electric systems (based on mesh- or node-analysis) is considered. The authors indicate the advantages of the simultaneous study of all four systems. *P. Franklin.*

**Braams, C. M. On the influence of friction on the motion of a top.** Physica 18, 503–514 (1952).

The author studies a spherical body, with an axially symmetric distribution of mass, spinning on a horizontal plane. Using elementary considerations, he discusses the effects on the motion of friction between the body and the plane. It appears that if the various parameters involved satisfy certain conditions, sliding friction causes the center of mass of the body to rise, and that this effect accounts for the behavior of the familiar toy called the "tippe top" or "toupie magique". *L. A. MacColl* (New York, N. Y.).

**Hugenholtz, N. M. On tops rising by friction.** Physica 18, 515–527 (1952).

This paper is similar to the one reviewed above, the only conspicuous difference being that here the mathematical theory is developed in a somewhat more complete and fundamental way. *L. A. MacColl* (New York, N. Y.).

**Stoppelli, Francesco. Sui fenomeni giroscopici in un solido qualsiasi.** Rend. Sem. Mat. Univ. Padova 21, 25–43 (1952).

The author considers a rigid body with a fixed point, subjected to external forces. It is assumed that initially the body is rotating with a large angular velocity $r_0$ about one of the axes of stable rotation passing through the fixed point. The subsequent motion of the body is discussed on the basis of the assumption that quantities of orders higher than the first in the small quantity $r_0^{-1}$ are negligible.

*L. A. MacColl* (New York, N. Y.).

**Metelicyn, I. I. On gyroscopic stabilization.** Doklady Akad. Nauk SSSR (N.S.) 86, 31–34 (1952). (Russian)

Stability criteria of motion of dynamical systems give conditions to be satisfied by the external forces in order that the roots of the characteristic equation have negative real parts. The well-known theorems of Lagrange, Kelvin, Routh and Žukovskiĭ are of this character. These theorems also reveal the influence of dissipative and gyroscopic forces on the stability of motion of conservative systems.

Since at present, however, the mechanical and electromechanical systems of interest are mostly nonconservative, the above mentioned criteria have been displaced by those of Hurwitz, Mihailovskiĭ, and Nyquist which are applicable to nonconservative systems. Nevertheless, these latter criteria are unable to answer questions of more general character, such as: (i) What is the influence of the dissipative, gyroscopic, or non-conservative forces on the stability of motion? (ii) Can the stability be improved or secured by addition of dissipative forces to those already acting, if the motion is unstable, and so on.

In the paper under review the author gives some theorems which might be of help in answering questions of this type, before one passes to a more detailed investigation of stability by the well-known standard methods.

Decomposing the generalized forces depending on the generalized velocities and those depending on the generalized coordinates only into dissipative forces $\beta_{ks}\dot q_k$ and gyroscopic forces $\gamma_{ks}\dot q_k$ and into conservative forces $\delta_{ks}q_k$ and properly nonconservative forces $\epsilon_{ks}q_k$ respectively, the equations of motion are

$$(1)\qquad a_{ks}\ddot q_s+\beta_{ks}\dot q_s+\gamma_{ks}\dot q_s+\delta_{ks}q_s+\epsilon_{ks}q_s=0\quad (k, s=1, 2, \cdots, n)$$

with $a_{ks}$, $\beta_{ks}$, $\delta_{ks}$ symmetric, $\gamma_{ks}$, $\epsilon_{ks}$ antisymmetric. Introduce the dissipation function $D$ and the potential energy $V$:

$$2D=\beta_{ks}\dot q_k\dot q_s,\quad 2V=\delta_{ks}q_kq_s.$$

On substituting $q_s=A_s\exp \mu t$ in (1) we have

$$(2)\quad [a_{ks}\mu^2+(\beta_{ks}+\gamma_{ks})\mu+(\delta_{ks}+\epsilon_{ks})]A_s=0\quad (k, s=1, 2, \cdots, n)$$

from which it follows that $\mu$ must be a root of the so-called characteristic equation (3) $\Delta(\mu)=0$. Assume that (3) has complex conjugate roots $\mu$ and $\mu'$; then to each such root there will correspond a system of complex conjugate coefficients $A_s$, $A_s'$. Suppose that the equations (2) are multiplied respectively by $A_1'$, $A_2'$, $\cdots$, $A_n'$, and added together. The resulting equation will be of the form

$$(4)\quad T(A, A')\mu^2+[D(A, A')+i\Gamma(A, A')]\mu \\ +V(A, A')+iE(A, A')=0$$

where $T, D, V, \Gamma$, and $E$ are real quantities, and $T$ is positive definite. Assume now the dissipative forces such that energy is being continually lost by the system, so that $D$ is a positive definite form. Then from equation (4) which differs from (3) only by some rearrangement of terms, it follows that the roots will have negative real parts if the condition $4TE^2-2D\Gamma E<D^2V$ is satisfied. From this condition of stability for nonconservative systems the author derives six theorems giving answers to some questions of the above mentioned type. *E. Leimanis* (Vancouver, B. C.).

## *Hydrodynamics, Aerodynamics, Acoustics*

**Byušgens, S. S. On stream-lines. II.** Doklady Akad. Nauk SSSR (N.S.) 84, 861–863 (1952). (Russian)

The first part [same Doklady (N.S.) 78, 837–840 (1951); these Rev. 13, 292] offered conditions for streamlines which still contained some cinematical elements. In this second paper the conditions are cast into a purely geometrical form: A necessary and sufficient condition that a given congruence can be taken as the congruence of streamlines of a stationary flow in an ideal incompressible liquid is 1) that the field of adjoint vectors $J$ of the given congruence must admit a family of orthogonal surfaces, 2) that the second adjoint field of vectors $J^*$ must be a gradient field. The second adjoint field is given by

$$J^*=J+\frac{\operatorname{rot} J_1\times J_2}{2(p_2-q_1)}.$$

*D. J. Struik* (Cambridge, Mass.).

**Payne, L. E. On axially symmetric flow and the method of generalized electrostatics.** Quart. Appl. Math. 10, 197–204 (1952).

A. Weinstein has pointed out recently that the flow about an axially symmetric body in ordinary three-dimensional space may be obtained from the electrostatic potential of the five-dimensional body of revolution possessing the same meridian profile [Trans. Amer. Math. Soc. 63, 342–354 (1948); these Rev. 9, 584]. In this paper A. Weinstein's method of "generalized electrostatics" is used to obtain the flow about a spindle and a lens. Using dipolar coordinates $\xi$, $\eta$ and the transformation $x+iy=ic\cot\frac{1}{2}(\xi+i\eta)$, the author gets the general potential function $\varphi(n)$ and the stream function $\psi(n)$ in the space of $n$ dimensions. The profile of a spindle is given by the curve $\xi=\xi_0$. If $\xi_0=\pi/2$, the spindle becomes a sphere and the well-known stream function for the sphere is obtained. Using the peripolar transformation $x+iy=-c\cot\frac{1}{2}(\xi+i\eta)$ the author gets the profile of a lens by the relations $\xi=\xi_1$, $\xi=\xi_2$ $(c>0)$ and the general potential function $\varphi(n)$ and stream function $\psi(n)$. The case $n=3$ was given by F. G. Mehler [J. Reine Angew. Math. 68, 134–150 (1868)].

It remains to be shown that this solution is unique. It is convenient to make use of an eigenvalue method due to Weinstein [Canadian J. Math. 1, 271–278 (1949); these Rev. 11, 65] which requires only a knowledge of the behavior of $\psi$ at infinity. By this method it can be shown that there is only one stream function $\psi$ which satisfies the prescribed conditions on the lens profile. If the stream function is unique, the potential $\Phi$ is also unique up to an arbitrary constant which must be zero in order that the potential vanish at infinity. Additional results concern the cases: sphere, two separated spheres, prolate spheroid, oblate spheroid, disc, and some internal problems. *M. Pinl.*

**Payne, L. E. A note on my paper, "On axially symmetric flow and the method of generalized electrostatics".** Quart. Appl. Math. 10, 398 (1953).

See the paper reviewed above.

**Loos, H. G. Tunnelwall interference for aerofoils and aerofoil-fuselage combinations in a tunnel with an octagonal section for incompressible flow.** Nationaal Luchvaartlaboratorium, Amsterdam. Report **A.1204**, A19–A27 (1951).

Consider a lifting wing symmetrically mounted in a tunnel of rectangular cross section $S$ with vertices $z=\pm\frac{1}{2}b\pm\frac{1}{2}hi$. Approximate the trailing vortex system by a dipole or by a vortex pair with velocity potentials (A) $i\mu/4\pi z$ or (B) $(\Gamma/4\pi i)\log[(z-c)/(z+c)]$, $c$ real. The perturbation velocities due to the tunnel walls can be obtained by superposing the velocities of image dipoles at $z_{mn}=nb+mhi$ or vortex pairs at $z_{mn}\pm c$, for integers $m$, $n$. Now remove congruent isosceles right triangles from each corner of $S$. G. K. Batchelor [Austral. Counc. Aeronaut. Rep. ACA-1 (1944); these Rev. 6, 78; also Rep. ACA-5 (1944)] has approximately cancelled the flow across the edges $E$ of slope $\pm1$ by adding to the rectangular tunnel solution vortex distributions with linearly varying strengths on $E$ and their images. If these are considered to have been approximated by discrete vortices at the midpoints of $E$ and their images, the author suggests approximating the four vortices nearest to $z_{mn}+\frac{1}{2}b+\frac{1}{2}hi$ by the quadrupole with potential $Q/8\pi(z-z_{mn}-\frac{1}{2}b-\frac{1}{2}hi)^2$, where $Q$ is chosen so the total flow across any segment $E$ vanishes. Similarly, for a midwing on a circular fuselage of radius $R$ in a rectangular tunnel, the author first reflects vortices (B) with respect to the circle to obtain a pair with potential

$$\text{(C)}\qquad (-\Gamma/4\pi i)\log[(z-2R^2/c)/(z+2R^2/c)],$$

and approximates (B) plus (C) by a dipole (A) of strength $\mu=2(c-R^2/c)\Gamma$. Form image-dipoles at $z_{mn}$, and reflect these with respect to the circle. Approximate the circular images by dipoles of equal strength at $z=0$ to obtain there an additional dipole of strength $\epsilon\mu$, $\epsilon$ real. Iterate to obtain a dipole of total strength $\mu/(1-\epsilon)$ at $z=0$ and its images at $z_{mn}$. In all cases the author indicates formal corrections for angle of attack and lift coefficient. *J. H. Giese.*

**Dolaptschijew, Bl. Anwendung der Methode von N. E. Kotschin zur Bestimmung des Gleichgewichtszustandes der zweiparametrigen Wirbelstrassen.** Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. 46, 357–368 (1950). (Bulgarian. German summary)

Considérons une rue de tourbillons formée de deux files rectilignes et parallèles de tourbillons ponctuels, régulièrement espacés sur chaque file, mais présentant un décalage quelconque d'une file à l'autre. Les conditions nécessaires de stabilité de cette configuration ont été étudiées antérieurement (entre autre par Kotchine et par l'A.) en particularisant la perturbation initiale. L'A. fait ici une discussion plus approfondie du problème, en envisageant des perturbations d'un type plus général. Il conclut à l'insuffisance du critère de stabilité classique: $\sin\alpha\pi=\sin\lambda\pi$ $(\alpha=h/l,\ \lambda=d/l$ ou $h$ est la distance des files, $l$ l'espacement des tourbillons sur celles-ci, $d$ le décalage des files): mais les configurations correspondantes seraient les "moins instables" au sens attaché à ce mot par l'A. *J. Kravtchenko* (Grenoble).

**Dolaptschijew, Blagowest. Verallgemeinertes Verfahren zur Stabilitätsuntersuchung beliebig geordneter Wirbelstrassen.** Annulaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 369–376 (1950). (Bulgarian. German summary)

En utilisant la méthode de Kotchine [cf. Kočin, Kibel', Roze, Hydrodynamique théorique (en russe), t. I, p. 207, Gostehizdat, Moscou-Leningrad, 1948] l'A. forme la condition nécessaire de stabilité d'une rue de tourbillons, constituée par deux files rectilignes et parallèles, les tourbillons sur chaque file étant équidistants, mais le décalage des files étant quelconque. Dans le cas particulier des configurations: 1) symétriques, 2) en quinconce ou alternés (v. Kármán), l'A. retrouve les conclusions de son travail antérieur [Spisanie Bulgar. Akad. Nauk **57**, 149–218 (1938)].

*J. Kravtchenko* (Grenoble).

**Kalinin, S. V. The discontinuous flow about an obstacle in the form of an arc of a second degree curve.** Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 966–984 (1950). (Russian)

L'A. se propose de former la solution du célèbre problème déterminé des sillages, symétrique, plan, en fluide indéfini, lorsque l'obstacle est un arc symétrique de section conique. La méthode utilisée est celle de Nekrasov [Izvestiya Ivanovo-Vosnesensk. Polytehn. Inst. no. **5** (1922)]; elle consiste à partir du système classique de Villat, vérifié par la fonction inconnue $T(\sigma)$ et par le paramètre inconnu $\lambda$: (1) $T(\sigma)=F[T(\sigma),\lambda]$, (2) $\lambda=\Phi[T(\sigma),\lambda]$, ou $F$ et $\Phi$ sont des fonctionnelles compliquées du type integro-differentiel non linéaire. L'A. se borne à résoudre par approximations successives l'equation (1), en supposant $\lambda$ constant, puis à déterminer approximativement et numériquement $\lambda$ en portant dans (2) la solution $T(\sigma,\lambda)$ de (1). Cette manière de faire ne semble pas à l'abri de toute reproche et A. Weinstein a depuis longtemps signalé les lacunes de ces raisonnements [cf. J. Kravtchenko, J. Math. Pures Appl. (9) **20**, 35–234 (1941), chap. III, par. 23; ces Rev. **3**, 219; **4**, 58]. J'ajoute que la démonstration de la convergence du processus d'approximations successives de Nekrasov paraît, à première vue, insuffisante sur quelques points.

En résumé, le présent travail, à notre avis, n'apporte pas de contribution au problème d'existence des solutions du problème déterminé des sillages. Au surplus, des résultats d'existence, recouvrant le cas des obstacles étudiés par l'A., sont connus depuis longtemps (Leray, Weinstein). Mais ces auteurs ont utilisé des raisonnements abstraits qui ne permettent pas le calcul approché de la solution cherchée. A cet égard, la méthode de l'A. offre de réels avantages et il est probable que, pour de petites valeurs de $\lambda$, il a formé de bonnes solutions approchées du problème déterminé. Signalons, enfin, que la bibliographie de l'A. ne fait état que des travaux publiés en langue russe. *J. Kravtchenko.*

**Fil'čakov, P. F. The hydrodynamic computation of a weir with two notches of unequal length.** Ukrain. Mat. Žurnal **2**, 92–109 (1950). (Russian)

Un problème de filtration des eaux souterraines conduit à chercher la représentation conforme sur un demi-plan d'un domaine infini à frontière polygonale, munie de coupures rectilignes. La solution indéterminée s'obtient aisément au moyen d'une formule analogue à celle de Schwarz-Christoffel. En utilisant la théorie des fonctions elliptiques, l'A. résout explicitement le problème déterminé, en effectuant le calcul numérique (des plus soignés) des paramètres intervenant dans la solution indéterminée. *J. Kravtchenko.*

**Storchi, Edoardo. Piccole oscillazioni dell'acqua contenuta da pareti piane.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 544–552 (1952).

The author shows how the method of an earlier paper [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **13**(**82**), 95–112 (1949); these Rev. **13**, 789] can be extended so that the function $\rho(x)$ (see the cited review) can be determined from the shape of the container. The method is applied to the case of standing waves on a sloping beach with slope $m\pi/2p$ [cf. Lewy, Bull. Amer. Math. Soc. **52**, 737–775 (1946); these Rev. **9**, 163]. The solutions do not seem to have physical significance for this problem, but when they are either symmetric or antisymmetric can be used to find wave motions in containers whose walls have slopes $\pm m\pi/2p$, not all combinations of $m$ and $p$ being now useful. *J. V. Wehausen* (Providence, R. I.).

**Sretenskiĭ, L. N. On a method of determination of waves of finite amplitude.** Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1952**, 688–698 (1952). (Russian)

The author treats gravity waves of finite amplitude in an infinitely deep perfect fluid by a method originally due to Stokes, but with what the author considers a more coherent exposition. If $z=x+iy$ denotes a point in the physical plane and $w=\varphi+i\psi$ is the complex velocity potential of the motion, it is assumed that they are related by

$$z=\frac{w}{c}-a_0 i-i\sum_{n=1}^{\infty}a_n e^{2\pi inw/c\lambda}.$$

The free surface condition, taken in the form $2gy=c^2-v^2$, and choice of axes so that the mean wave height is zero give equations for determination of the $a_n$. The novelty is in the organization of this rather elaborate calculation. Values of $a_n$ up to $n=6$ are calculated and the corresponding relation between velocity $c$, wave-length $\lambda$, and a parameter associated with amplitude is given. The author does not prove convergence of various expansions, stating that this follows from work of Nekrasov (unfortunately unavailable).

*J. V. Wehausen* (Providence, R. I.).

**Sekerž-Zen'kovič, Ya. I. On the three-dimensional problem of standing waves of finite amplitude on the surface of a heavy liquid.** Doklady Akad. Nauk SSSR (N.S.) **86**, 35–38 (1952). (Russian)

The author extends his treatment of standing waves of finite amplitude to the case of doubly periodic waves in infinitely deep water [cf. same Doklady (N.S.) **58**, 551–553 (1947); Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **15**, 57–73 (1951); these Rev. **10**, 646; **12**, 870]. As in the cited papers, the author states that he is able to prove existence and uniqueness of the waves for not too large amplitudes. An approximate formula for the shape of the free surface agrees with that given recently by Penney and Price [Philos. Trans. Roy. Soc. London. Ser. A. **244**, 254–284 (1952), pp. 278–281; these Rev. **13**, 790].

*J. V. Wehausen* (Providence, R. I.).

**Biésel, F. Equations générales au second ordre de la houle irrégulière.** Houille Blanche **7**, 372–376 (1 insert) (1952).

The author considers the motion resulting from the superposition of a finite number of simple harmonic gravity waves. The computations (which are not included) were carried out through the second order and are presented on a folded insert. Pertinent formulas are given in both Lagrangian and Eulerian coordinates. In order to elucidate the

nature of the interaction between two waves of different length, the author considers in more detail the motion resulting from the superposition of two waves of the same amplitude and direction but with slightly differing length (i.e., the case of beating). Formulas are also given on the insert for the analogous case of an "imperfect" standing wave when the two waves move in opposite directions. [Cf. Sekerž-Zenkovič, Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1951**, no. 5, 68–83; these Rev. **13**, 396.]
*J. V. Wehausen* (Providence, R. I.).

**Slezkin, N. A. Differential equations for the motion of pulp.** Doklady Akad. Nauk SSSR (N.S.) **86**, 235–237 (1952). (Russian)

L'A. étudie le phénomène de transport des particules solides du sol dans l'eau. Il forme d'abord un système d'équations aux dérivées partielles pour chaque composante du mélange. L'ensemble des relations obtenues est inférieur au nombre d'inconnues. Des hypothèses simplificatives appropriées conduisent à représentent les lois du phénomène au moyen d'un système de huit équations aux dérivées partielles entre huit fonctions inconnues. *J. Kravtchenko.*

**Kuznecov, M. D. Hydrodynamics of an eccentric ring-shaped cross-section.** Doklady Akad. Nauk SSSR (N.S.) **85**, 715–717 (1952). (Russian)

The author considers the generalization of Poiseuille flow in which the viscous fluid flows in the region between two parallel circular cylinders, the two not being necessarily coaxial. Since his fundamental equation assumes axial symmetry, the solution is in error except in the coaxial case, which is well known. In particular, his anomalous result that the resistance of a pipe can be decreased by inserting a small rod near the wall is incorrect.
*J. V. Wehausen* (Providence, R. I.).

**Comolet, Raymond. Ecoulement radial d'un fluide compressible visqueux entre deux plans parallèles.** C. R. Acad. Sci. Paris **235**, 1190–1193 (1952).

**Valensi, Jacques, et Clarion, Claire. Oscillations amorties d'une sphère dans un fluide visqueux.** C. R. Acad. Sci. Paris **235**, 1097–1099 (1952).

**Gadd, G. E. Some hydrodynamical aspects of the swimming of snakes and eels.** Philos. Mag. (7) **43**, 663–670 (1952).

The author studies the motion of idealized models of snakes and eels executing transverse sinusoidal undulations of variable amplitude. Using a strip method and other assumptions, he estimates the forces on the bodies and their propulsive efficiency while swimming. *D. Gilbarg.*

**Lighthill, M. J. On the squirming motion of nearly spherical deformable bodies through liquids at very small Reynolds numbers.** Comm. Pure Appl. Math. **5**, 109–118 (1952).

This paper investigates the self-propulsion of small spherical bodies induced by deformations of their shape. As in the previous work of Taylor on the motion of tailed micro-organisms [Proc. Roy. Soc. London. Ser. A. **209**, 447–461 (1951); **211**, 225–239 (1952); these Rev. **13**, 596; **14**, 104], the mathematical details are based on neglect of the inertial terms in the Navier-Stokes equations. The author's approximate solutions of the axially symmetric flow equations under boundary conditions determined by given oscillatory deformations of the body show that the mean velocity of these self-induced translations is of the order of the square of the amplitude of the deformations. In three illustrative examples in which the maximum surface strain is 1/3, the author studies the efficiency of the motion, and shows that "even in the most efficient of the three, the mean power required to obtain a given mean velocity is twenty times that given by the Stokes' formula for the uniform motion of a rigid sphere under an external force". *D. Gilbarg.*

**Tatsumi, Tomomasa. Stability of the laminar inlet-flow prior to the formation of Poiseuille régime. I, II.** J. Phys. Soc. Japan **7**, 489–495, 495–502 (1952).

The first part of this paper deals with the calculation of the velocity distribution in a circular pipe before the parabolic velocity profile is established. This type of profile is shown, in the second part, to have a stability characteristic similar to that in the plane Poiseuille flow. The instability of the pipe flow is therefore ascribed to this part of the flow field. A minimum critical Reynolds number of about $10^4$ is found at a point 17 times the radius of pipe downstream from its entrance. *C. C. Lin* (Cambridge, Mass.).

∗**Heisenberg, W. On the stability of laminar flow.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 292–296. Amer. Math. Soc., Providence, R. I., 1952.

This paper gives a concise discussion of the classical problem of the stability of parallel flows, with special emphasis on the case of plane Poiseuille motion, which was first shown to be unstable by the author [Ann. Physik (4) **74**, 577–627 (1924)]. Several points in the theory are suggested for further clarification and a brief history of the problem, particularly in the nineteen-twenties, is described. Finally, the author discusses the physical mechanism for causing the instability and its relation with fully developed turbulence. *C. C. Lin* (Cambridge, Mass.).

**Chandrasekhar, S. The invariant theory of isotropic turbulence in magneto-hydrodynamics.** Proc. Roy. Soc. London. Ser. A. **204**, 435–449 (1951).

The invariant theory of isotropic turbulence in magneto-hydrodynamics is developed analogously to the usual theory. The conductivity of the fluid is taken to be effectively infinite and there is supposed to be no external electro-magnetic field. Invariant integrals of Loitsiansky's type are derived for the three double correlations of velocity and magnetic field. Those associated with magnetic field are shown to be identically zero. The energy equations are derived exhibiting the dissipation of kinetic energy by viscosity, magnetic energy by conduction, and the exchange of energy in the two forms. Finally, the author considers the case where a stationary condition is maintained by a constant supply of kinetic energy to the system. It is concluded that the magnetic energy is essentially associated with eddies having large wave-numbers. *C. C. Lin* (Cambridge, Mass.).

**Chandrasekhar, S. The invariant theory of isotropic turbulence in magneto-hydrodynamics. II.** Proc. Roy. Soc. London. Ser. A. **207**, 301–306 (1951).

The author continues his development of the theory of isotropic turbulence in magneto-hydrodynamics [see the paper reviewed above] to include the correlations of total pressure with fluctuations of velocity and magnetic field. These involve correlations of the fourth order in the fluctuations of velocity and magnetic field. They are then broken

down in terms of correlations of the second order by using relations valid for joint normal distribution of the fluctuations. These relations are then expressed in terms of simpler relations among the defining scalars. *C. C. Lin.*

**Lee, T. D. On some statistical properties of hydrodynamical and magneto-hydrodynamical fields.** Quart. Appl. Math. **10**, 69–74 (1952).

The Fourier transforms of the standard equations of magneto-hydrodynamics [cf. Batchelor, Proc. Roy. Soc. London. Ser. A. **201**, 405–416 (1950); these Rev. **11**, 699; and Chandrasekhar, see the two papers reviewed above] are written down and the equivalent in the wave number ($k$) space of the law of conservation of energy which obtains in the absence of viscosity and conductivity is interpreted as meaning that the spectral law of the kinetic energy ($\overline{u^2}$) and the magnetic energy ($\overline{H^2}$) are both governed by the density of free modes which exist in the $k$-space, namely, $4\pi k^2dk$. From a heuristic discussion of the case when dissipation by viscosity and conductivity takes place, the author concludes that in the limit of large Reynolds numbers both $\overline{u^2}$ and $\overline{H^2}$ follow the $k^{-5/3}$-Kolmogoroff law. [The author's discussion ignores the fact that $\overline{u^2}$ admits a Loitsiansky invariant independently of the growth of a small initial magnetic field by the stretching of the lines of force, and that therefore there can be no transfer of energy from the velocity to the magnetic fields for $k\to 0$; from this last circumstance one can conclude that the spectrum of $\overline{H^2}$ for $k\to 0$ must be as $k^6$ in contrast to $k^4$ which obtains for $\overline{u^2}$.] *S. Chandrasekhar.*

**Sedov, L. I. On the general theory of one-dimensional motions of a gas.** Doklady Akad. Nauk SSSR (N.S.) **85**, 723–726; addendum, **87**, 4 (1952). (Russian)

Various relations have been collected for use in determining approximate solutions or asymptotic behavior of solutions of unsteady one-dimensional or cylindrically or spherically symmetrical flow problems. For example, near $r=0$, the density, velocity, and pressure are expanded in series

$$\rho(r,t)=r^s\sum_0^\infty \omega_n(t)r^{n-1},\quad u(r,t)=\sum_0^\infty \varphi_n(t)r^{2n+1},$$

$$p(r,t)=p_0(t)+r^s\sum_0^\infty \psi_n(t)r^{2n+2},$$

where $s>0$, and $p_0(t)$ or $\omega_0(t)$ is arbitrary, and the rest of $\omega_n$, $\varphi_n$, and $\psi_n$ are determined recursively by an infinite system of differential equations. Also, behind a shock at $r_2(t)$ propagating into undisturbed gas in which the speed of sound is $a_1$, $\partial u/\partial r$, $\partial\rho/\partial r$, and $\partial p/\partial r$ are known functions of $r_2$, $q=a_1^2(dr_2/dt)^{-2}$, and $dq/dr_2$. The author suggests that some unspecified integral relations can be used to determine $q(t)$ and $\omega_0(t)$ by methods analogous to those of Rayleigh-Ritz or Galerkin in elasticity. He also corrects the equations for $\partial u/\partial r$, $\partial\rho/\partial r$, and $\partial p/\partial r$ at the shock to take viscosity and heat conduction into account. *J. H. Giese.*

**Kawamura, Ryuma. Reflection of a wave at an interface of supersonic flows and wave patterns in a supersonic compound jet.** J. Phys. Soc. Japan **7**, 482–485 (1952).

The paper considers the waves reflected and transmitted when a weak wave falls upon the interface between two supersonic streams. It is found that the transmitted wave is always of the same type as the incident. The results are applied to a two-dimensional supersonic jet discharging into a supersonic stream at a slightly different pressure. The ultimate pressure in the jet is the original pressure of the outer stream. The change is brought about by monotonic pressure changes at the successive wave reflections at the boundary of the jet, or by alternating compressions and expansions, depending upon the ratio of the Mach numbers in the jet and outside. This particular jet problem has also been examined recently by S. I. Pai [J. Aeronaut. Sci. **19**, 61–65 (1952)]. *D. C. Pack* (Manchester).

**Kawamura, Tōru. On the detached shock wave in front of a body of revolution moving with supersonic speeds.** J. Phys. Soc. Japan **7**, 486–488 (1952).

The method of Dugundji [J. Aeronaut. Sci. **15**, 699–705 (1948); these Rev. **10**, 494] is applied to find the position of the detached shock wave in front of a sphere. Comparison is made with some experimental results and with some calculations made by the author in a previous paper [Mem. Coll. Sci. Univ. Kyoto. Ser. A. **26**, 207–230 (1950)] in which he fitted an incompressible potential flow (approximately) between the sphere and the detached shock wave.

*D. C. Pack* (Manchester).

**Travers, Serge. Limitation des gradients, et de leur dérivée logarithmique dans les ondes de choc, par les formules de Chapman.** C. R. Acad. Sci. Paris **235**, 1099–1101 (1952).

**Blank, Helga. Applicazione del metodo di Ritz al calcolo della corrente compressibile attorno ad un cilindro circolare.** Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **321**, 11 pp. (1951).

The variational method of Ritz [J. Reine Angew. Math. **135**, 1–61 (1908)] is applied to the non-linear partial differential equation for the velocity potential of the compressible, inviscid, non-circulatory, two-dimensional flow about a circular cylinder. The mixed flow is treated, i.e., the flow is partly subsonic and partly supersonic. This treatment is based on the variational formulation of this problem by Braun [Ann. Physik (5) **15**, 645–676 (1932)]. A numerical example is included. *C. Saltzer* (Cleveland, Ohio).

**Mitchell, A. R., and McCall, Francis. The rotational field behind a bow shock wave in axially symmetric flow using relaxation methods.** Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 371–380 (1952).

**Kaplan, Carl. On transonic flow past a wave-shaped wall.** NACA Tech. Note no. **2748**, 43 pp. (1952).

In an earlier investigation of this problem [same Tech. Notes no. **2383** (1951); these Rev. **12**, 875], the author showed that in the transsonic range, the perturbation of the potential function from the uniform flow is approximately proportional to a function $f$ which satisfies the equation $f_{yy}=f_xf_{xx}$. Subject to the boundary conditions $\partial f/\partial x=-1$, $\partial f/\partial y=0$ at $y=\infty$; $\partial f/\partial y=-k\sin x$ at $y=0$, $-\infty<x<\infty$, he solves this in the form

$$(*)\qquad f=-x+\sum_{n=1}^\infty f_n\sin(nx),$$

where

$$f_n=\sum_{p=0}^\infty e^{-(n+2p)y}P_{n,p}(k,y),$$

where $P_{n,p}(k,y)$ is a polynomial in $y$ but not in $k$. Expanding the $P_{n,p}$ in double series, non-linear recurrence formulae for their coefficients are obtained. The author solves several of these by the method of generating functions and is able to

calculate a substantial number of the lower order coefficients. The dependence of the convergence of the series (*) upon $k$ is discussed partially. *E. Pinney.*

**Tumašev, G. G. Construction of channels and nozzles with a given distribution of the subsonic velocities along the walls.** Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk 1, 47–50 (1948). (Russian)

The equations for steady plane irrotational flow can be written as $\partial s/\partial\psi=-K^{\frac{1}{2}}\partial\beta/\partial\varphi$, $\partial\beta/\partial\psi=-K^{\frac{1}{2}}\partial s/\partial\varphi$, where

$$K=(1-\lambda^2)[1-\lambda^2(\gamma-1)/(\gamma+1)]^{(1+\gamma)/(1-\gamma)},$$
$$\lambda ds/d\lambda=\{(1-\lambda^2)/[1-\lambda^2(\gamma-1)/(\gamma+1)]\}^{\frac{1}{2}},$$

$\beta$ is the inclination of the velocity relative to the $x$-axis, $\lambda$ is the speed of flow in units of the critical speed of sound, $\varphi$ is the velocity potential, and $\psi$ the stream function. For small $\lambda$ the author replaces $K$ by 1, and then $s-i\beta$ is an analytic function of $\varphi+i\psi$. For an infinite channel with $s$ prescribed as a function of arclength $\epsilon$ along the wall, the problem reduces to determining $s-i\beta$ with given boundary values $s=s(\varphi)$ on the edges of the strip $0\leqq\psi\leqq\frac{1}{2}\pi$. The channel walls are given by $x+iy=\int e^{i\beta}d\epsilon$ on $\psi=0$ or $\frac{1}{2}\pi$. For a finite channel joining two regions of parallel flow the problem reduces to determining $s-i\beta$ on a rectangle with given boundary values $s(\varphi)$ on $\psi=0$ or $\omega_2$, and $\beta=$constant on $\varphi=\pm\frac{1}{2}\omega_1$. *J. H. Giese* (Havre de Grace, Md.).

**Tumašev, G. G. Construction of a nozzle according to the distribution of supersonic velocities along the walls.** Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk 2, 133–134 (1950). (Russian)

For the stream and velocity potential functions of non-linearized plane supersonic flow Christianovich [Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 215–222 (1947); also Amer. Math. Soc. Translation no. 10 (1950); these Rev. 9, 390; 11, 272] has developed approximations involving two arbitrary functions of single characteristic variables. The author has applied these to determine the boundaries of a channel from a knowledge of the velocity as a function of one characteristic variable on a characteristic $AB$ and the speed as a function of arclength on the streamlines through $A$ and $B$. He also shows that this problem can be solved by a graphical method of characteristics. *J. H. Giese.*

**Phythian, J. E. Some unsteady motions of a slender body through an inviscid gas.** Quart. J. Mech. Appl. Math. 5, 301–317 (1952).

Linearized potential theory is applied to solve the problem of unsteady motion of slender bodies in both the subsonic and supersonic flight regimes. Bodies of revolution and bodies of more general cross-section are treated for the cases of non-uniform forward velocity and oscillatory motion. Equations are derived for the drag, lateral forces, and moments. The analysis is closely allied to Ward's [same J. 2, 75–97 (1949); these Rev. 10, 644] for steady flight. Again the equation of motion reduces to Laplace's equation in two dimensions for every plane perpendicular to the flight direction. However, determination of the pressure and forces for the unsteady problems is considerably more complicated than for the steady cases.

Let $l$ denote the body length, $f(t)$ the forward displacement of the body, $\alpha(t)$ the angle of incidence, and $t$ the time. The author shows that the aerodynamic forces associated with the unsteady motions are small compared to the corresponding steady forces if $lf''(t)\ll(f'(t))^2$, $l^2\alpha''(t)\ll\alpha(t)(f'(t))^2$, and $l\alpha'(t)\ll\alpha(t)f'(t)$. *W. R. Sears* (Ithaca, N. Y.).

**Stewartson, K. On the linearized potential theory of unsteady supersonic motion. II.** Quart. J. Mech. Appl. Math. 5, 137–154 (1952).

The author develops an extension of Evvard's method in steady flow [NACA Tech. Note no. 1382 (1947); these Rev. 8, 610] that leads to an iterative solution of the unsteady flow problem. He remarks that incorrect extensions had been given previously by Evvard [ibid. no. 1699 (1948); these Rev. 10, 78] and Stewart and Li [Quart. Appl. Math. 9, 31–45 (1951); these Rev. 13, 86]. The method being impractical for wings having no supersonic portion of leading edge, a method analogous to that applied to the rectangular wing by Galin and Magnaradze [see Galin, Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 465–474 (1947)=Tech. Rep. no. F-TS-1217-IA (GDAM A9-T-36). Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio (1949); these Rev. 9, 254; 11, 273] is developed for the delta wing by introducing a set of pseudo-polar coordinates in the space associated with the hyperbolic distance $(x^2-y^2-z^2)^{1/2}$. This formally reduces the problem of the unsteady flow past a delta wing to an equivalent problem in steady flow. Unfortunately, the solution of the latter appears (to the reviewer) to be quite formidable, so that it is not clear that the author's ingenious reduction achieves any real simplification of the original problem.
*J. W. Miles* (Los Angeles, Calif.).

**Goodman, Theodore R. The quarter-infinite wing oscillating at supersonic speeds.** Quart. Appl. Math. 10, 189–192 (1952).

The title problem is solved via Gardner's method, the details of the analysis being similar to those of the author's solution for the gust problem [J. Aeronaut. Sci. 18, 519–526 (1951); these Rev. 13, 298]. The results agree with those obtained by the reviewer [ibid. 16, 381 (1949); same Quart. 9, 47–65 (1951); these Rev. 10, 755; 12, 649] and K. Stewartson [Quart. J. Mech. Appl. Math. 3, 182–199 (1950); these Rev. 12, 453] but not with those of H. J. Stewart and T. Y. Li [J. Aeronaut. Sci. 17, 529–539 (1951); these Rev. 12, 453]. The reviewer remarks that the errors in Stewart and Li's method [same Quart. 9, 31–49 (1951); these Rev. 13, 86] recently have been discussed in some detail by Stewartson [see the preceding review]. *J. W. Miles.*

**Kemp, Nelson H. On the lift and circulation of airfoils in some unsteady-flow problems.** J. Aeronaut. Sci. 19, 713–714 (1952). See ~~at~~ the corresponding note p. 1279

The work of v. Kármán and Sears [same J. 5, 379–390 (1938)] and Sears [J. Franklin Inst. 230, 95–111 (1940); these Rev. 2, 28] is extended to obtain (1) the circulation about an airfoil in sinusoidal motion and (2) the lift due to a sinusoidal gust having a phase velocity that is not equal to the flight velocity. A numerical table is included. The results have application to unsteady flow in a compressor. Contrary to the author's statement that his parameter $\mu$ can be an arbitrary complex number, the reviewer believes that it must be restricted by the boundary conditions at infinity. *J. W. Miles* (Los Angeles, Calif.).

**Leslie, D. C. M. Supersonic theory of downwash fields.** Quart. J. Mech. Appl. Math. 5, 292–300 (1952).

The previous work of (A) Lagerstrom and Graham [Douglas Aircraft Co. Rep. SM-13007 (1947)], (B) Robinson and Hunter-Tod [Coll. Aeronaut. Cranfield. Rep. no. 10 (1947); these Rev. 9, 479], and (C) Mirels and Haefeli [NACA Tech. Note no. 1925 (1949)] is reviewed and com-

pared. The results are rederived using Hadamard's solution to the wave equation, and the approximation of (C) is improved and simplified. The author restricts his work to a rectangular wing but states that it is easily extended, using, in part, results from (B). In view of the later work of Lagerstrom and Graham [Douglas Aircraft Co. Rep. SM-13743 (1950)] and Spreiter and Sacks [J. Aeronaut. Sci. 18, 21–32, 72 (1951); these Rev. 13, 594], of which the author evidently was unaware, his assertions that "linearized theory gives good results for thin wings at small angles of incidence" and that "viscous effects should be considered before those of non-linearity" would appear to be unwarranted.

*J. W. Miles* (Los Angeles, Calif.).

**Miles, John W. A note on the damping in roll of a cruciform winged body.** Quart. Appl. Math. 10, 276–277 (1952).

The author applies Ward's slender-body theory to the calculation of the damping in rolll of a slender body of revolution bearing cruciform wings. The potential is obtained through conformal mapping of the cross-sectional profile of the cruciform winged body to a circle. Then the integral of rolling moment is solved. In the limiting case, the solution checks with some known results.

*C. C. Chang* (Manchester).

**Wasserman, Robert H. Theory of supersonic potential flow in turbomachines.** NACA Tech. Note no. 2705, 44 pp. (1952).

The author considers the very difficult problem of calculating the purely supersonic flow of an ideal gas through a turbomachine with a finite number of blades. The velocity potential is described by a quasi-linear hyperbolic partial differential equation of the second order in three independent variables. It is proposed to solve this problem by iterated integration along characteristic surfaces (Picard's method), the initial conditions being given along some non-characteristic surface. The mathematical problem is discussed and the solution procedure set up explicitly in a form adaptable to numerical methods.

To illustrate the application of this scheme, the author considers the "design" of a single row of rotating blades, confining the analysis to a one blade passage. It is assumed that the gas enters tangentially to the suction surface of the blade so that a shock wave proceeds only from the leading edge of the pressure side of the blade to some (undetermined) point on the suction side of the adjacent blade. The shock divides the flow fluid, giving one problem ahead of the shock and one downstream, to be matched at the shock wave. The initial surface of the upstream problem consists of the shock (of undetermined position) and the portion of the suction side of the blade which lies ahead of the shock. The initial surface for the downstream problem consists of the shock wave and the portion of the suction side of the blade lying downstream of the shock. In order to treat these problems within the "frame work" of this analysis, the rather unrealistic assumption is made that the flow is irrotational both ahead and behind the shock. With sufficient conditions prescribed on the suction side of the blade, these two problems may be solved numerically for the velocity potential in the field—which incidentally determines the shape of the pressure side of the blade. Presumably the prescription on the suction surface of the blade must be altered until an admissible shape for the pressure side is achieved. It is not clear just how much information may be prescribed on the inner and outer shrouds without "overdetermining" the solution.

Although no calculations are reported the author states that, with the problem arranged for a high speed computing machine, he does not expect the work to be prohibitive. Unfortunately, the final appraisal of this approach must be withheld until at least one such solution is produced.

*F. E. Marble* (Pasadena, Calif.).

***Ostroumov, G. A. Svobodnaya konvekciya v usloviyah vnutrennei zadači. [Free convection in cases of internal problems.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 256 pp. (27 plates). 10.90 rubles.

This monograph represents a survey of some fundamental aspects of free convective heat transfer for internal flow. For steady low velocity flow the linearized biharmonic equation is applied with various boundary conditions of temperature and velocity distribution to flow primarily in vertical circular channels. In a later chapter the case of non-linear steady laminar flow as arising in such applications as free convection in horizontal channels is analyzed by introducing a linearizing parameter and solving as successive approximations the resulting set of linear equations. Conventional experimental methods of observing free convective flows are described in several chapters. No discussion of non-steady or turbulent flow is given and no reference is made to the many semi-empirical correlations of practical value. Much of the material is taken directly from recent articles appearing in the Russian technical literature.

*N. A. Hall* (Minneapolis, Minn.).

**Halatnikov, I. M. Heat exchange between a solid body and Helium II.** Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 22, 687–704 (1952). (Russian)

Vengono esaminati i possibili meccanismi per lo scambio di calore fra un corpo solido e l'elio II. Si mostra che il fenomeno avviene mediante l'irraggiamento e assorbimento di oscillazioni alla superficie di separazione. Lo scambio di energia è proporzionale alle differenza delle quarte potenze delle temperature. Viene poi calcolato il coefficiente di trasmissione del secondo suono attraverso a una sottile lamina conduttrice del calore. *G. Toraldo di Francia.*

**Halatnikov, I. M. The hydrodynamics of solutions of foreign particles in Helium II.** Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 23, 169–181 (1952). (Russian)

Si deducono le equazioni dell'idrodinamica delle soluzioni concentrate di particelle estranee nell'elio II. Si dimostra che tali particelle prendono parte soltanto al moto normale per qualsiasi valore della concentrazione, purchè appartengano a un corpo che allo stato puro non presenta superfluidità. Vengono dedotte le equazioni dell'idrodinamica dell'elio II (formalmente valide per qualsiasi valore delle velocità). (Riassunto del autore.) *G. Toraldo di Francia* (Firenze).

**Kronig, R., and Thellung, A. On the hydrodynamics of non-viscous fluids and the theory of helium II.** Physica 18, 749–761 (1952).

L'idrodinamica classica di un fluido perfetto nel caso di moto irrotazionale viene messa sotto forma canonica prendendo per densità lagrangiana lo scostamento della pressione dal suo valore di equilibrio e per variabili coniugate il potenziale di velocità e la densità. Il formalismo e le notazioni sono identici a quelli usati da Wentzel [Einführung in die Quantentheorie der Wellenfelder, Deuticke, Wien, 1943;

questi Rev. 9, 556] e con identico metodo avviene la quantizzazione. La densità di hamiltoniana viene sviluppata in serie di potenze delle variabili canoniche sull'esempio di Landau e Khalatnikov [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 19, 637–650, 709–726 (1949)]. Se si ritiene soltanto il primo termine (quadratico) della hamiltoniana, si ottiene un gas di fononi non interagenti. L'autore discute l'eccitazione termica del fluido in vista delle applicazioni all'elio II. *G. Toraldo di Francia* (Firenze).

**Smolyakov, P. T. On the reduction of the equations of motion in the atmosphere to ones integrable by quadratures.** Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk 1, 75–78 (1948). (Russian)

Il s'agit de résoudre les équations du mouvement dans le plan d'une particule d'air soumise aux forces de Coriolis et de viscosité, les accélérations étant supposées nulles. En introduisant une variable complexe $s=v-iu$ ($u$ et $v$ étant les composantes de la vitesse du vent), on est ramené à une équation différentielle linéaire du second ordre:

$$\frac{d^2s}{dz^2}+A\frac{ds}{dz}+Bs=C,$$

$z$ étant l'altitude; $A$ et $B$ sont fonctions du coefficient de viscosité cinématique $\mu$ et de la densité $\rho$; $C$ est fonction de $\mu$ et $\rho$ et du gradient de pression. L'auteur introduit un changement des variables $\psi=\psi(z)$ de façon à rendre les coefficients $A$ et $B$ constants. On trouve aisément que $\psi$ doit satisfaire à l'équation $\psi=m^{-1}\int(\rho/\mu)^{1/2}dz$ avec la condition $d(\mu\rho)^{1/2}/dz=n/m$ ($m$, $n$ des constantes réelles). L'auteur étudie différents cas particuliers: (a) $\mu=$cte; (b) $\rho=$cte; et (c) $\rho=\rho_0e^{-2az}$. *M. Kiveliovitch* (Paris).

**Smolyakov, P. T. On the stationarity of a baric field.** Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk 2, 93–99 (1950). (Russian)

L'auteur étudie le mouvement d'une particule d'air en négligeant la composante verticale et dans l'hypothèse du vent géostrophique. En utilisant l'équation de I. A. Kibel', on voit que le mouvement est dans ce cas stationnaire et non divergent (voir au sujet des équations de Kibel' notre mémoire [J. Sci. Météorologie 1, 72–74 (1949); ces Rev. 11, 280]). L'auteur étudie en détail ce mouvement.

*M. Kiveliovitch* (Paris).

**Charney, J. G. On the scale of atmospheric motions.** Geofys. Publ. Norske Vid.-Akad. Oslo 17, no. 2, 17 pp. (1948).

The purpose of the paper is to find what approximations are permissible in the numerical integration of the meteorological equations. Typical magnitudes are assigned to horizontal and vertical distances, the static stability, wave velocity, and wind velocity, for large scale motions. From these quantities, Charney shows the following. 1) The acceleration is an order of magnitude smaller than the Coriolis and pressure gradient forces, permitting the geostrophic approximation in most equations not involving divergence. 2) Vertical advection is generally smaller than horizontal advection. 3) The horizontal divergence contains two nearly equal terms of opposite sign. 4) The density change in the equation of continuity is small compared to the other terms in the equation. Essentially, horizontal divergence is equal and opposite to vertical divergence. 5) The hydrostatic equation represents a satisfactory approximation to the vertical equation of motion. *H. Panofsky.*

**Sretenskiĭ, L. N. Propagation of waves from a sounding disc.** Moskov. Gos. Univ. Učenye Zapiski 154, Mehanika 4, 275–285 (1951). (Russian)

Vengono dedotte varie espressioni asintotiche del noto integrale

$$\Phi=\frac{v}{2\pi}\int_S\int\frac{e^{-ikR}}{R}dS$$

nel caso che la lunghezza d'onda sia molto piccola rispetto al diametro del disco. *G. Toraldo di Francia* (Firenze).

## *Elasticity, Plasticity*

**Ling, Chih-Bing, and Yang, Kuo-Liang. On symmetrical strain in solids of revolution in spherical co-ordinates.** J. Appl. Mech. 18, 367–370 (1951).

This paper presents the expressions for the displacements and stresses in the spherical coordinates, in terms of a stress function, for a solid of revolution in the state of symmetrical strain. Such expressions are useful in dealing with solids of revolution, which consist of spherical boundaries. The expressions are applied, as an illustration, to find the stresses in a large tension member having a spherical cavity. The required stress function is constructed in terms of Legendre polynomials. The stresses obtained agree with known results. (From the author's summary.) *R. M. Morris.*

**Schmidt, Kurt. Behandlung ebener Elastizitätsprobleme mit Hilfe hyperkomplexer Singularitäten.** Ing.-Arch. 19, 324–341 (1951).

L. Sobrero [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 6, 1–64 (1934); Theorie der ebenen Elastizität . . . , Teubner, Leipzig, 1934] introduced a method for handling two-dimensional elasticity problems based on the determination of a hypercomplex analytic function $f(z)=\sigma_y+jb+j^2\sigma_x+j^3\tau$ where $z=x+jy$ and $1+2j^2+j^4=0$ (this method is the analogue of handling two-dimensional hydrodynamical problems by the determination of an analytic function $f(z)=u-iv$, where $u$ and $v$ are the velocity components, and $i^2+1=0$). The present paper contains an exposition of Sobrero's method and its application to the determination of the corresponding hypercomplex function $f(z)$ in the following cases: (a) concentrated load in an infinite plane; (b) concentrated load in a half-plane; (c) concentrated load in an infinite strip; (d) two diametrically opposite concentrated loads (one at the center) in the interior of a circle. The results are compared with some previous results in the literature. *J. B. Diaz* (College Park, Md.).

**Legendre, Robert. Elastostatique plane.** Recherche Aéronautique no. 27, 3–5 (1952).

The author has developed a method for treating problems in plane elasticity by using "doubly complex" functions [Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 42 (1951); these Rev. 14, 333]. Recently, K. Schmidt [see the paper reviewed above] has applied a method of hypercomplex functions, due to L. Sobrero [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 6, 1–64 (1934)] to solve certain problems in plane elasticity. In the present note the author compares these two methods, and finds that Sobrero's hypercomplex functions are related to his doubly-complex functions, and that these last mentioned functions are simply one-parameter families of ordinary analytic func-

tions, considered only for infinitely small values of the parameter. As an example of the facility of handling doubly-complex functions, the author considers the problem of determining the stress state in a circular disk with two concentrated opposing forces at two of its points (unlike Schmidt, in the paper mentioned above, the center of the circle need not be a point of application of one of the forces).

*J. B. Diaz* (College Park, Md.).

**Karunes, B. On the distribution of initial stress due to dislocation in an infinite plate containing two unequal circular holes.** Indian J. Phys. **26**, 317–328 (1952).

Using bi-polar coordinates the author obtains the Airy stress function for two problems on the state of initial stress due to dislocation in an infinite plate containing two unequal circular holes. The dislocations treated are (1) a parallel fissure joining the two circular holes or joining each hole to infinity and (2) two opposite kinds of wedge-shaped fissures of the same apex angles on two sides of the $x$-axis. Having found suitable stress functions to give these dislocations, the author has to add others which give no dislocation, to cancel out the stresses which these functions yield in each case on the boundaries of the holes and at infinity. Cases of equal holes in each of these problems are treated as particular examples, and graphs showing the hoop stress round the holes are drawn for two particular equal hole boundaries.

*R. M. Morris* (Cardiff).

***Funk, P., und Berger, E. Eingrenzung für die grösste Durchbiegung einer gleichmässig belasteten eingespannten quadratischen Platte.** Beiträge zur angewandten Mechanik. Federhofer-Girkmann-Festschrift, pp. 199–204. F. Deuticke, Vienna, 1950.

The method by which the problem in the title is dealt with relates to ideas of E. Trefftz [Math. Ann. **100**, 503–521 (1928)], K. Friedrichs [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1929**, 13–20], J. B. Diaz and H. J. Greenberg [J. Math. Physics **27**, 193–201 (1948); these Rev. **10**, 213], and C. Weber [Z. Angew. Math. Mech. **22**, 130–136 (1942); these Rev. **5**, 81] and consists in employing minimum principles for the deflection $u$ of a clamped plate, which satisfies

$$\Delta\Delta u = q, \text{ in } D; \quad u = \frac{\partial u}{\partial n} = 0, \text{ on } C,$$

where $C$ is the boundary of $D$, in order to bound the values of $u$ at any preassigned point of $D$. The authors formulate these minimum principles in physical terms and make a numerical application to a uniformly loaded clamped square plate ($D$ is the set $-1<x, y<1$ and $q=1$) and obtain that $0.01999<u(0, 0)<0.02051$, which is to be compared with the approximate value 0.02032 given by W. Ritz [J. Reine Angew. Math. **135**, 1–61 (1908), esp. p. 43].

*J. B. Diaz* (College Park, Md.).

**Reissner, Eric. A problem of finite bending of circular ring plates.** Quart. Appl. Math. **10**, 167–173 (1952).

We consider axi-symmetrical transverse bending of a circular ring plate under the action of transverse edge forces. Let $a$ be the inner radius of the plate and $a+b$ the outer radius. If $b$ is sufficiently small compared with $a$ and if the conditions of support permit it, it is generally considered allowable to obtain deflections and stresses in the ring plate by beam theory rather than by plate theory. The purpose of the present note is a more detailed analysis of this problem. We shall show that the question of whether beam theory and plate theory give essentially the same results depends not only on the ratio $b/a$ but also on the magnitude of the deflections of the plate. Briefly, let $\phi_0$ be a quantity of the order of magnitude of the deflection of the plate according to beam theory divided by the width $b$ of the plate, and let $\mu$ be a dimensionless parameter of the form $[12(1-\nu^2)]^{1/2}b^2/ah$, where $\nu$ is Poisson's ratio and $h$ the thickness of the plate. We find that beam theory is applicable as long as $\phi_0\mu \ll 1$. When $\phi_0\mu = O(1)$, then plate theory must be used and the appropriate equations are those of finite bending. We find further that when $\phi_0\mu \gg 1$ a boundary layer effect is encountered. In order to have a continuous transition between finite bending of beams and finite bending of plates we must use a system of equations for bending of plates which is more general than the equations of Kirchhoff and von Kármán for small finite plate bending. A more general system of this nature, applicable to axi-symmetrical bending of circular plates has recently been given [E. Reissner, Proc. Symp. Appl. Math., vol. 1, Amer. Math. Soc., New York, 1949, pp. 213–219; vol. 3, McGraw-Hill, New York, 1950, pp. 27–52; these Rev. **11**, 286; **12**, 557]. (From the author's summary.)

*L. M. Milne-Thomson* (Sevenoaks).

**Reissner, Eric. Pure bending and twisting of thin skewed plates.** Quart. Appl. Math. **10**, 395–397 (1953).

In the elementary theory of thin rectangular plates, two cases of pure bending arise when uniformly distributed bending moments $M_1$ and $M_2$ are applied to opposite pairs of edges, and also when oppositely directed pairs of loads are applied at diagonal corners of the plate. The author provides similar cases of pure bending and torsion of thin skewed (parallelogram) plates and gives the effect of this skewness on the bending and torsional rigidity of plates. In the case of torsion of skew plates the deflection $w$ is given by

$$w(x, y) = k_1[xy - k_2(y^2 - \sigma x^2)]$$

where $k_2$ involves the angle of skewness and vanishes for rectangular plates and $k_1$ is the same constant for both types of plates. An immediate result is that a skew plate has a smaller torsional rigidity than a non-skew plate. In pure bending the deflection is

$$w(x, y) = k_3[x^2 - \sigma y^2 - k_4 xy]$$

where $k_3$ is the same constant for both types and $k_4$ vanishes in rectangular plates. The constants $k_2$, $k_4$ involve the angle of skewness and $k_1$, $k_3$ are independent of it. Skew plates have a smaller bending stiffness than non-skew plates and at the same time are accompanied by a torsional deformation. A leading factor ½ is missing from the last equation of the paper.

*D. L. Holl* (Ames, Iowa).

**Clark, R. A., Gilroy, T. I., and Reissner, E. Stresses and deformations of toroidal shells of elliptical cross section with applications to the problems of bending of curved tubes and of the Bourdon gage.** J. Appl. Mech. **19**, 37–48 (1952).

Starting from the small deflection theory of thin shells of revolution subjected to rotationally symmetric loading [see E. Reissner, Reissner Anniversary Volume, pp. 231–247, Edwards, Ann Arbor, Mich., 1948; these Rev. **11**, 69], the authors solve three problems of toroidal shells having elliptical cross-sections. These problems are: (a) closed shell under uniform normal wall pressure; (b) open shell under end bending moments; (c) combination of (a) and (b) to solve the Bourdon tube problem. The stresses and deformations are found to be functions of the two parameters $bc/ah$

and $b/c$, $b$ and $c$ being the semi-axes of the elliptical cross-sections, $a$ the distance from the axis of revolution to the center of the cross-section, and $h$ the wall thickness. For small values of $bc/ah$, it is found that trigonometric series solutions are suitable, while, for large values of $bc/ah$, the differential equations are solved by asymptotic integration. Numerical results are presented graphically as functions of $bc/ah$ for several values of $b/c$. *H. D. Conway.*

**Stepanov, R. D., and Šerman, D. I. Torsion of a circular bar weakened by two longitudinal cylindrical circular cavities.** Akad. Nauk SSSR. Inženernyĭ Sbornik **11**, 127–150 (1952). (Russian)

The Saint Venant torsion problem for circular cylinders weakened by longitudinal circular cylindrical cavities was previously considered by G. M. Goluzin [Mat. Sbornik **41**, 246–276 (1934)] and Chih-Bing Ling [Quart. Appl. Math. **5**, 168–181 (1947); these Rev. **8**, 612]. The method of solution proposed by Goluzin is based on an algorithm of successive approximations, the effectiveness of which has not been tested on special problems. Ling reduces the problem to the solution of an infinite system of linear equations and obtains an approximate solution by truncating the system. Numerical examples considered by Ling suggest that his method may not lead to satisfactory results when the radius of one of the cavities exceeds the shortest distance from the exterior boundary to the boundary of the cavity. The authors of this paper solve the problem stated in the title by constructing an auxiliary density function defined on the boundary of the triply connected domain under consideration. The complex torsion functions is then determined from the density function by integration. The density function is shown to satisfy a certain integral equation, which is solved approximately by reducing it to a system of algebraic equations. The authors do not give a rigorous justification of the approximations, but rather crucial examples selected by them suggest that their method is effective when other methods are expected to fail. The essential ideas of this paper are based on considerations previously employed by Šerman [Doklady Akad. Nauk SSSR (N.S.) **63**, 499–502 (1948); Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1951**, 969–995; these Rev. **10**, 651; **13**, 301] in solving similar problems for doubly connected domains. *I. S. Sokolnikoff* (Los Angeles, Calif.).

**Sakadi, Zyurô. Criticism on the equations of flexural vibration of a thin bar.** Math. Japonicae **2**, 79–85 (1951).

The usual equation (1) for flexural vibrations of rods, and (2) the more complicated equations taking into account rotary inertia and (3) this and also shear distortion are considered. The velocity of sinusoidal waves $y=\exp[i(pt+\gamma x)]$ is determined, and (2) and (3) give results differing from (1) by terms of the same order. The complete three-dimensional elastic theory [see Love, Elasticity, 4th ed., Cambridge, 1927, p. 291] is used for a circular rod by expansion of the resulting Bessel functions in series, and the difference from (1) is of the same order as the additional terms due to (2) and (3), but differing from them. (3) is closer to the correct solution, but since the error is of the same order as the correction from (1), it is considered not to provide a satisfactory higher order theory. *E. H. Lee.*

**Koiter, W. T. On Grammel's linearisation of the equations for torsional vibrations of crankshafts.** Nederl. Akad. Wetensch. Proc., Ser. B. **54**, 464–467 (1951).

In the theory of torsional vibrations of crankshafts, the connecting rod and piston appear as a variable inertia term which is a periodic function of the angular position. Small torsional oscillations $\theta$ superposed on uniform rotation are considered. The variable term mentioned above introduces non-linearity. Expansion to first order in terms of $\theta$ and its time derivatives gives equations differing from those used previously [see Biezeno and Grammel, Technische Dynamik, Springer, Berlin, 1939], since certain terms were formerly omitted, so that the earlier work would be expected to give an incorrect assessment of the influence of variable inertia. The solution is sought through a perturbation expansion in terms of a small coefficient representing the fluctuating part of the inertia, and to first order the added terms cancel out and do not modify the previous result. *E. H. Lee* (Providence, R. I.).

***Kammerer, Albert. Les propriétés mécaniques des solides réels et la théorie de l'élasticité.** Actualités Sci. Ind., no. 1161. Hermann et Cie., Paris, 1951. 132 pp. 1125 francs.

In the introduction the author states his aims as follows: 1) to review the results of mechanical tests on various materials; 2) to derive simple mathematical formulas describing these results; 3) to use a generalized version of these formulas to broaden the foundations of the theory of elasticity. The present review is restricted to the items of mathematical interest. The author begins by establishing the fundamental equations for a linear visco-elastic solid with two elastic constants and two coefficients of viscosity. No reference is made to the considerable body of classical and recent literature on this subject [see, for instance, T. Alfrey, Mechanical behavior of high polymers, Interscience, New York, 1948; these Rev. **10**, 169]. Next, the idea is introduced that an initially isotropic material may become orthotropic under strain even within the elastic range. It is assumed that Young's modulus for each principal direction of strain depends in a linear fashion on the corresponding principal strain and the cubical dilatation. It is claimed that, within a certain range at least, the influence of temperature is adequately represented if these strain terms include the thermal expansion. The resulting formulas are applied to simple tension, flexure, and torsion; the algebraic work is simplified by the startling assumption that only the change of volume gives rise to viscous stresses but not the change of shape. For a specimen suddenly subjected to a tensile stress $\sigma$ that is kept thereafter constant, two critical stresses $\sigma'$ and $\sigma''$ are obtained: for $\sigma<\sigma'$, the extension remains finite; for $\sigma'<\sigma<\sigma''$, the extension grows indefinitely within a finite period of time; for $\sigma>\sigma''$, even the instantaneous extension accompanying the application of the load becomes infinite. Similar critical loads are obtained for flexure and torsion. In discussing the behavior of a solid subjected to a homogeneous state of stress, the author drops the viscous terms to achieve greater mathematical simplicity. Again critical conditions of loading are determined that lead to infinite strains. In the opinion of the reviewer such attempts at obtaining the condition of rupture by extrapolating laws of elasticity far beyond their range of validity cannot be justified by physical arguments. The same remark applies to the author's theory of endurance. *W. Prager.*

***Prager, William, and Hodge, Philip G., Jr. Theory of perfectly plastic solids.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. x+264 pp. $5.50.

The authors restrict their considerations to perfectly plastic materials, that is, materials in which the stresses in

the plastic regime satisfy a yield condition characterized by the vanishing of a certain invariant of the stress tensor. Such a choice has been made because this field of plasticity has reached the most definitive form. In addition to a wide selection of solved problems, the authors also point out directions for possible future research. It is proposed that the book can be used as a text for a senior or first year graduate course in mechanics or applied mathematics. Mathematical techniques, beyond this level, which may be required are supplied. The only exception is a short application of complex variable techniques.

The major topics considered are: basic concepts of stress, strain, and stress-strain relations; trusses and beams; torsion of cylindrical or prismatic bars; plane strain including problems with axial symmetry, general theory, specific problems, contained plastic deformations, limit analysis; and finally, general extremum principles. Each chapter concludes with a rather complete bibliography and an extensive selection of problems.

Special commendation should be given for the choice of problems. A number of the problems in each set are examples of text material. Other exercises are problem versions of research papers not included in the main body of the work. Corresponding references are given. The remaining problems extend given results and anticipate, by particular examples, general principles which are established later. The last two types of problems include such questions as obtaining the St. Venant-Mises material from the Prandtl-Reuss laws, three-dimensional stress-space analysis, characteristics in the torsion problem, torsion of a circular bar of variable cross-section, shakedown of trusses, geometrical properties of Hencky-Prandtl nets, limit design of plane frames and beams.

The presentation is extremely lucid in great contrast to a number of earlier papers in this field. Many of the difficulties encountered are discussed with thoroughness. In particular, the relation of the Prandtl-Reuss law and the St. Venant-Mises law in contained plastic flow and fully plastic flow is brought out in detail. This distinction is further clarified as a consequence of limit analysis. The recent work of Drucker, Greenberg, and Prager on extremum principles and limit analysis is clearly set forth. The authors offer suggestions of problems and methods for future research using these new techniques.

On the whole, the book should prove to be a delightful introduction to the field of plasticity as well as a rather thorough survey of this particular phase of the subject. It is hoped that other aspects of plasticity will appear in the future in as well written a form.

*G. H. Handelman.*

**Drucker, D. C., Greenberg, H. J., and Prager, W. The safety factor of an elastic-plastic body in plane strain.** J. Appl. Mech. 18, 371–378 (1951).

In this paper, the authors discuss the theory of the safety factor in plane strain for a material which is in contained plastic deformation. The theory (developed in an appendix) is applied to a body whose cross-section consists of an inner circular boundary and an outer square boundary, both with a common center. Due to the delay in publication of this article, the theory has been discussed in greater detail in a text [see the book reviewed above]. *N. Coburn.*

**Craggs, J. W. The normal penetration of a thin elastic-plastic plate by a right circular cone.** Proc. Roy. Soc. Edinburgh. Sect. A. 63, 359–370 (1952).

The paper is concerned with the propagation of elastic and plastic deformation in an initially unstressed diaphragm of infinite extent that is hit by a rigid cone moving with uniform velocity normal to the diaphragm. Perfectly plastic and strain-hardening diaphragms are considered, and the yield condition and flow rule of v. Mises are used. Depending on the cone angle and speed, the solutions fall into four classes. For large cone angles and low speeds, the diaphragm remains flat and elastic up to the cone. There are discontinuities in slope and radial stress where the diaphragm reaches the cone. For smaller cone angles or higher speeds, there is a flat plastic region between the outer elastic part and the bulged part of the diaphragm. As the speed is further increased, the width of the bulged region is reduced until finally the bulge disappears and the diaphragm remains flat except where it is in contact with the cone. For a perfectly plastic diaphragm and large values of the cone angle and speed, a fourth type of solution arises: the diaphragm remains flat except where it is in contact with the cone; where the diaphragm meets the cone, there is a shearing stress in the diaphragm. *W. Prager.*

**Franciosi, Vincenzo. Sul calcolo a rottura delle strutture monodimensionali in regime elasto-plastico.** Giorn. Genio Civile 90, 387–400 (1952).

The first part of this paper surveys the mechanical behavior of elastic-plastic structures [following Prager and Symonds, Proc. Symposia Appl. Math., v. 3, McGraw-Hill, New York, 1950; these Rev. 12, 563] and methods of limit analysis [following Greenberg and Prager, Grad. Div. Appl. Math. Brown Univ. Tech. Rep. A18-1 (1949); these Rev. 11, 559]. The second part discusses reduction of limit moments by axial forces. An iterative method of limit analysis is suggested, using axial forces obtained in one step to compute reduced limit moments for the next step. The influence of axial forces on limit loads is small for structural frames of conventional design but significant for trusses with rigid joints. *W. Prager* (Providence, R. I.).

# MATHEMATICAL PHYSICS

## *Optics, Electromagnetic Theory*

**Mikaèlyan, A. L. On a method of solution of the inverse problem of geometric optics.** Doklady Akad. Nauk SSSR (N.S.) 86, 933–936 (1952). (Russian)

L'autore chiama problema inverso dell'ottica geometrica quello di trovare tutte le distribuzioni di indice di rifrazione che danno luogo a determinate traiettorie per i raggi. Nel caso piano egli dimostra che, se nel mezzo $n_1(x, y)$ i fronti d'onda sono dati da $S_1(x, y)$ = costante, gli stessi fronti d'onda si avranno nel mezzo $n(x, y) = n_1(x, y)S[S_1(x, y)]$, essendo $S$ una funzione arbitraria dell'argomento $S_1(x, y)$.

*G. Toraldo di Francia* (Firenze).

**Grümm, H. Zur Frage der geometrisch-optischen Intensitätsverteilung im Bildraum von Elektronenlinsen.** Optik 9, 281–311 (1952).

A continuazione di un lavoro di Glaser e Grümm [Optik 7, 96–120 (1950); questi Rev. 12, 305] vengono calcolate le isofote nell'immagine data da una lente elettronica. L'ap-

prossimazione è quella dell'ottica geometrica e si considerano soltanto le aberrazioni di Seidel. Per la misura sperimentale della caustica l'autora studia l'ombra di un filo rettilineo, apparentemente senza rendersi conto che tutti i dettagli del metodo sono notissimi da molti anni.

*G. Toraldo di Francia* (Firenze).

**Muscia, Calogero. Studio di una lente elettronica con il metodo W. K. B.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 575–582 (1952).

The author considers the case of a magnetic field given by

$$H_z=H_0\left[1+\left(\frac{z}{a}\right)^2\right]^{-3/2}.$$

The corresponding differential equation for the paraxial rays by an appropriate substitution of variables can be given the form

$$v''(\varphi)+[(1+k^2)-k^2\cos^2\varphi]v(\varphi)=0.$$

This is a Mathieu equation. Applying the W.K.B. method one can find an approximate solution involving the elliptic integrals of the second kind. Several numerical calculations have been carried out and the results are represented in a graphical form. *G. Toraldo di Francia* (Florence).

**Foix, Auguste. Sur la plus simple solution, en coordonnées polaires, des équations de Maxwell, et ses applications aux formules de la réfraction vitreuse dans les lentilles, et aussi sur le principe de Huygens.** J. Phys. Radium (8) 13, 445–450 (1952).

L'autore afferma, con argomenti non del tutto chiari che, per metter ordine nei fondamenti dell'ottica fisica è opportuno partire da quella che egli chiama la più semplice soluzione delle equazioni di Maxwell in coordinate polari.

*G. Toraldo di Francia* (Firenze).

**Sturrock, P. A. The imaging properties of electron beams in arbitrary static electromagnetic fields.** Philos. Trans. Roy. Soc. London. Ser. A. 245, 155–187 (1952).

Riferendosi a coordinate curvilinee, chiamando $z$ la coordinata lungo il raggio e $u$, $v$ due coordinate perpendicolari ad essa, il principio di Fermat dell'ottica elettronica viene scritto nella forma $\delta\int m dz=0$, essendo $m=m(u, v, u', v', z)$ una funzione determinata dalla forma del campo elettromagnetico. Essa puo venire sviluppata in una serie di polinomi $m^{(r)}$ di ordine $r$ rispetto a $u$, $v$, $u'$, $v'$ più una piccola variazione cromatica, caratterizzata da un parametro $\epsilon$. Viene effettuato il calcolo e la discussione dei vari termini di questo sviluppo fino alle aberrazioni del secondo ordine.

*G. Toraldo di Francia* (Firenze).

**Kleinwächter, Hans. Die Berechnung ebener Elektronenbahnen in speziellen elektrischen und magnetischen Feldern mittels komplexer Ortsvektoren.** Arch. Elektr. Übertragung 6, 315–318 (1952).

Il metodo enunciato nel titolo non ha bisogno di ulteriori chiarificazioni e viene applicato a un gran numero di casi particolari in cui il campo elettrico giace in un piano e dipende dal punto e dal tempo, mentre il campo magnetico è perpendicolare al piano stesso e dipende solo dal tempo.

*G. Toraldo di Francia* (Firenze).

**Ahiezer, A. I., and Sitenko, A. G. On the passage of a charged particle through an electron plasma.** Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 23, 161–168 (1952). (Russian)

La funzione di distribuzione delle posizioni $\mathbf{r}$ e delle velocità $\mathbf{v}$ nel plasma obbedisce all'equazione differenziale

$$\frac{\partial F}{\partial t}+\mathbf{u}\frac{\partial F}{\partial \mathbf{r}}+\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}[\mathbf{uH}]\right)\frac{\partial F}{\partial \mathbf{u}}=0$$

mentre $\mathbf{E}$ e $\mathbf{H}$ obbediscono alle equazioni di Maxwell, nelle quali si tiene conto della presenza del plasma e di una particella estranea di carica $q$ e di velocità $v$. Ponendo $F=f_0+f$, dove $f_0$ è la distribuzione maxwelliana e ammettendo che $f$ sia molto piccola, in modo da poterne trascurare il quadrato, l'autore studia la soluzione del sistema di equazioni e ne ricava l'interazione della particella col plasma.

*G. Toraldo di Francia* (Firenze).

**Frank, Ernest. Electric potential produced by two point current sources in a homogeneous conducting sphere.** J. Appl. Phys. 23, 1225–1228 (1952).

The electric potential produced by positive and negative point current sources located in a homogeneous conducting sphere is obtained for arbitrary source locations and separations. Certain special cases of the general solution are also presented. *From the author's summary.*

**Smith-White, W. B. The Poisson-Kelvin hypothesis and the theory of dielectrics.** J. Proc. Roy. Soc. New South Wales 85 (1951), 82–112 (1952).

The author points to the discrepancies between various force and torque relations used in the theory of dielectrics and traces the origin of the inconsistencies to a misinterpretation of the Poisson-Kelvin hypothesis, which replaces a polarized dielectric by a surface-charge density $\sigma=P_n$, and space-charge density $\rho=-\operatorname{div}\mathbf{P}$ for definition of the potential function. He rather would hypothesize a volume force $\mathbf{F}=(\mathbf{P}\cdot\nabla)\mathbf{E}$ and a volume torque $\mathbf{G}=\mathbf{P}\times\mathbf{E}$ together with a surface tension $\mathbf{T}=\frac{1}{2}\mathbf{P}\cdot(\mathbf{E}_+-\mathbf{E}_-)\mathbf{n}$ as fundamental statements and from these compute the total mechanical actions on the dielectric, considering the potential function as auxiliary.

To demonstrate his viewpoint, the author first considers a discrete charge $e$ for which the force action is $e\mathbf{E}$ and a point dipole of moment $\mathbf{p}$ for which the force action is $(\mathbf{p}\cdot\nabla)\mathbf{E}$, the torque $\mathbf{G}=\mathbf{p}\times\mathbf{E}$. He calls attention to the fact that although electric charge could not be created or destroyed, dipole moment is creatable and destructible, which implies that the general electrostatic system need not be mechanically conservative and he later shows examples. Expanding to assemblies of discrete point charges and point dipoles, a general work formula is derived

$$\Delta W=-\tfrac{1}{2}\Delta\sum e\phi+\tfrac{1}{2}\sum(\mathbf{p}\cdot\Delta\mathbf{E}-\mathbf{E}\cdot\Delta\mathbf{p})$$

which is demonstrated to hold for a polarized dielectric if one makes the hypotheses stated above, i.e., defines volume force and torque in direct analogy to the mechanical actions on the discrete elements. The author then shows that the concepts of electrostatic energy and of mechanical work function have to be examined carefully and illustrates inconsistencies by references to the literature and detailed discussions. *E. Weber* (Brooklyn, N. Y.).

**Breitenhuber, L. Über einige streng integrierbare Fälle elektromagnetischer Koppelschwingungen zweier Hohlraumresonatoren.** Acta Physica Austriaca **5**, 45–68 (1951).

The author gives some applications of the method developed by Ledinegg and Urban [same Acta **4**, 180–196 (1950); these Rev. **12**, 777] to determine the natural frequencies of two weakly-coupled resonators. He considers the Mathieu function solutions of Maxwell's equations in elliptic cylinder coordinates, and determines the unperturbed and perturbed frequencies for a compound resonator consisting of the two halves of an elliptic cylinder separated along the major axis and coupled through a small gap. The limiting case of two semicircular cylindrical resonators, coupled through a small hole at the center of their flat faces, is discussed in detail. *M. C. Gray* (Murray Hill, N. J.).

**Agostinelli, Cataldo. Sulla propagazione di onde elettromagnetiche in un tubo conduttore riempito di dielettrico eterogeneo.** Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 331–347 (1951).

This paper continues the development of the theory of propagation of waves in cylindrical conductors filled with nonhomogeneous dielectric, as originated by D. Graffi [Ann. Mat. Pura Appl. (4) **30**, 233–239 (1949); these Rev. **12**, 65]. The author discusses the form of Maxwell's equations for an orthogonal coordinate system $(q_1, q_2, z)$, where the transverse coordinates represent two orthogonal systems of isothermals, i.e., the two-dimensional Laplacians of $q_1$ and $q_2$ vanish, while $(\text{grad } q_1)^2 = (\text{grad } q_2)^2 = Q^{-2}$. He considers, in particular, a circular cylinder for which appropriate coordinates are $q_1 = \log r$, $q_2 = \theta$, with $Q = r^2$. It is shown that waves of electric type or of magnetic type may be propagated independently as long as $\epsilon$ is a function of $r$ only. There are also waves of more general type, $E(z) = \varphi(r) \sin(n\theta + \alpha)$, $H(z) = \psi(r) \cos(n\theta + \alpha)$ ($n$ an integer), the solution in this case being obtained from a fourth order linear differential equation. *M. C. Gray* (Murray Hill, N. J.).

**Dänzer, H. Über elektrische und akustische Einschwingvorgänge.** Ann. Physik (6) **10**, 395–412 (1952).

L'autore si riferisce per chiarezza al caso elettrico delle oscillazioni intrattenute mediante un accoppiamento induttivo. La tensione ai terminali dell'oscillatore può essere espressa nella forma $V(t) = \sum_{-\infty}^{+\infty} \beta_k \varphi_k(t) e^{p_k t}$ dove $p_k$ sono i poli della resistenza complessa e $\beta_k$ i corrispondenti residui. Seguono allora le equazioni differenziali per le $\varphi_k$,

$$d\varphi_k/dt = I(t) e^{-p_k t},$$

essendo $I(t)$ la corrente ai terminali dell'oscillatore. Viene esaminato in particolare il caso in cui la relazione fra $I(t)$ e $V(t)$ è lineare. *G. Toraldo di Francia* (Firenze).

**De Donder, Th. Sur les équations électromagnétiques de Maxwell.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 693–694 (1952).

The author derives a modified form of the Maxwell field equations from a variational principle by adjoining the Lorentz condition as a side condition and using a Lagrange multiplier. *A. H. Taub* (Urbana, Ill.).

***Sommerfeld, Arnold. Electrodynamics. Lectures on theoretical physics, Vol. III.** Translated by Edward G. Ramberg. Academic Press, Inc., New York, N. Y., 1952. xiii+371 pp. $6.80.

This book presents a very clear exposition of Maxwell's electromagnetic field theory with extensions for the special theory of relativity and the electrodynamics of moving media on the basis of Minkowski's original treatment. The illustrative applications of the theory are mainly restricted to what the author calls "summation problems" as distinct from the more advanced "boundary-value problems" which are included in vol. VI of the series, "Partial differential equations of physics" [Academic Press, New York, 1949; these Rev. **10**, 608].

Part I gives the fundamental concepts of Maxwell's field theory with stress upon the proper assignment of four fundamental dimensions, adding charge to the three mechanical dimensions, length, mass, and time. The extended rationalized Giorgi Unit System MKSC is employed throughout. The axiomatic foundation of Maxwell's theory is seen in Faraday's law of induction and in Ampère's law of the magnetic circulation (for Maxwell's total current), both stated in integral form. The supplementary axioms are the non-existence of true magnetism (Hertz) and the continuity equation of electricity (Kirchhoff). On this basis, the differential equations and the boundary conditions are derived. Stress is laid upon the fact that **B** should be the magnetic field strength responsible for force action analogous to **E** for the electric field.

Part II gives a concise derivation of the phenomena from the Maxwell equations, treating briefly polarization, magnetization, the Poynting energy-flow concept, the Hertzian vector, and retarded potentials. The skin effect and wave propagation along a single cylindrical conductor are treated in considerable detail (an original contribution by the author [Ann. Physik (3) **67**, 233–290 (1899)]). Then follows a basic discussion of general plane wave propagation with reference to cylindrical wave guides and to dielectric wave guides. Again considerable detail is given for the wave propagation along two parallel cylindrical conductors (first solved by G. Mie [ibid. (4) **2**, 201–249 (1900)]).

Part III introduces the concept of the four-dimensional potential functions and vector operations and stipulates isotropy in space and time, thus defining the Maxwell equations as invariant in the four-dimensional manifold. From this, the Lorentz transformation is then deduced, the Lorentz contraction and Einstein's dilatation of time. Applications are shown to the theory of the single electron in uniform and accelerated motion, and the relativistic particle mechanics. The electromagnetic theory of the electron is sketched in the last section.

Part IV gives Minkowski's equations of moving media, the consequences and experimental verifications (Roentgen current, Rowland effect). The final sections review the approaches to the generalization of Maxwell's equations so as to include the theory of the elementary particles, and to Einstein's general theory of relativity and the unified field theory.

*E. Weber* (Brooklyn, N. Y.).

**Ghosh, C. S., and Venkata Rao, P. The primitive machine of Kron.** J. Indian Inst. Sci. Sect. B. **34**, 123–139 (1952).

Starting with the equations of a rotating electrical machine in which the windings are at right angles in space, the authors derive the equations of a shaded-pole motor in which the stator windings are at an arbitrary angle. The new equations are established by means of a matrix of transformation that correlates the spatial configurations of the two sets of windings.

*G. Kron* (Schenectady, N. Y.).

**Morgan, Samuel P., Jr. Mathematical theory of laminated transmission lines. II.** Bell System Tech. J. 31, 1121–1206 (1952).

The author continues the calculations of an earlier paper [same J. 31, 883–949 (1952); these Rev. 14, 338] wherein he discussed the propagation of electromagnetic waves through a homogeneous, isotropic dielectric bounded first by two parallel, plane impedance sheets (laminated parallel plate waveguide) and then by two coaxial, cylindrical impedance sheets (laminated coaxial transmission line). In this paper for each of these two structures the author discusses the propagation problem when the homogeneous, isotropic dielectric occupying the region between the laminated conductors, i.e., the "main dielectric", is replaced by material consisting of alternate layers of conducting and insulating media. *C. H. Papas* (Pasadena, Calif.).

**Haacke, Wolfhart. Über die freien Schwingungen in $n$-fachen Netzwerken mit pulsierenden Parametern.** Arch. Elektr. Übertragung 6, 114–119 (1952).

Electrical networks composed of constant inductances and variable (with time) capacitances are considered. The capacities are all supposed to have the form const.$\times(1+h(t))^{-1}$ where $h(t)$ is a periodic function. The mesh equations are then $Lq''+K(1+h(t))q=0$ where $L$ and $K$ are constant matrices. Using a linear transformation which simultaneously diagonalizes $L$ and $K$, the problem is reduced to a number of Hill equations. *E. N. Gilbert.*

**Zadeh, L. A. A general theory of linear signal transmission systems.** J. Franklin Inst. 253, 293–312 (1952).

The function space representation of signals and of linear devices which operate on signals is discussed in detail and is offered as a means of unifying the various common approaches (Fourier transform, impulse response, etc.) to linear transmission theory. *E. N. Gilbert.*

**Ryder, Frederick L. Network analysis by least power theorems.** J. Franklin Inst. 254, 47–60 (1952).

Analogies of the theorem of least work and Castigliano's theorem are formulated for linear alternating current networks. [Related questions were considered by G. A. Campbell [Trans. Amer. Inst. Elec. Engrs. 30, 873–913 (1911)].] *R. J. Duffin* (Pittsburgh, Pa.).

**Darlington, Sidney. Network synthesis using Tchebycheff polynomial series.** Bell System Tech. J. 31, 613–665 (1952).

A method is developed for finding a function of the frequency which approximates a given gain (or phase) function on an interval of the frequency axis. The approximating function can be realized exactly as the gain of a finite network of jumped elements. The procedure is to expand both the given and the approximating functions in a series of Tchebycheff polynomials appropriate to the interval. By the introduction of a new variable these series are related to power series on the unit circle. The approximations are made relative to these power series and leads to a sort of least square error. In some special cases it is feasible to modify the method so as to obtain an approximation in the Tchebycheff sense. A number of practical examples show the utility of the method for design. *R. J. Duffin.*

**Tellegen, B. D. H. A general network theorem, with applications.** Philips Research Rep. 7, 259–269 (1952).

Let branch currents $i$ and branch voltages $v$ satisfy Kirchhoff's first and second laws respectively, that is, $\sum i=0$ at every node and $\sum v=0$ around every mesh. As a consequence it is shown that (*) $\sum iv=0$ where the summation is over all branches of the network. The energy and reciprocity relations are then deduced from this theorem. If given arbitrary voltages are applied to a $2n$-pole, it is shown that the difference between electric and magnetic energy will depend only on the admittance matrix of the $2n$-pole and not on the particular network used for realizing it. [In a paper by R. Bott and the reviewer (Trans. Amer. Math. Soc. 74, 99–109 (1953)) the relation (*) is interpreted in vector space language as defining orthogonal complementary subspaces of current and voltage.] *R. J. Duffin.*

**van Est, W. T., and Overbeek, J. Th. G. Electrokinetic effects in a network of capillaries. I, II.** Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 347–356, 357–362 (1952).

A network is considered whose branches are capillary tubes. The steady viscous flow of a fluid through such a network is governed by the hydraulic analogs of Kirchhoff's laws. If the fluid is also an electrical conductor, such as an electrolyte, an electrokinetic network results. It has long been known that an interaction exists between steady electric and hydraulic flow [M. Smoluchowski, Bull. Internat. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. 1903, 182–199, esp. p. 184]. The authors replace Ohm's law for a branch by the relations

$$(*) \qquad Le+Dp=i, \quad De+Fp=v.$$

Here $e$ and $p$ are the electric and hydraulic pressure drops across the branch; $i$ and $v$ are the electric and hydraulic currents through the branch. The constants $L$, $F$, and $D$ may be termed electric, hydraulic, and mutual conductances of the branch. A clear and precise analysis of the equations governing the flow in an electrokinetic network is given. This analysis is carried out on the junction point basis; it results that the network equations can be expressed in the form (*) with the following interpretation: $L$, $F$, and $D$ are matrices, $e$ and $p$ are vectors whose components give junction point electric and hydraulic pressures, and $i$ and $v$ are vectors whose components give the electric and hydraulic currents leaving the network at the junction points. An expression is derived for the mutual conductance $D_j$ across two junctions, other junctions being electrically and hydraulically insulated. In particular, if the constants $L$, $F$, and $D$ have ratios which do not change from branch to branch, it results that $D_j$ is proportioned to the ordinary joint conductance between these junction points. The factor of proportionality is such that there is no anomalous effect termed "the apparent $\zeta$ depression". *R. J. Duffin.*

## *Quantum Mechanics*

**Jordan, P. Zur axiomatischen Begründung der Quantenmechanik.** Z. Physik 133, 21–29 (1952).

A general discussion along the lines of the author's earlier work with von Neumann and Wigner, in the course of which he associates the existence of an elementary length with the question of whether there exists an algebraic system satisfying all the postulates for a Jordan algebra, except the distributivity, but admitting real scalars and being formally real. *I. E. Segal* (Chicago, Ill.).

**Wigner, E. P. Die Messung quantenmechanischer Operatoren.** Z. Physik 133, 101–108 (1952).

Conservation principles are used to show that not all hermitian operators can be measurable, except possibly as limiting cases, and it is mentioned that a similar conclusion follows from considerations of relativistic invariance.

*I. E. Segal* (Chicago, Ill.).

**Aleksandrov, A. D. On Einstein's paradox in quantum mechanics.** Doklady Akad. Nauk SSSR (N.S.) 84, 253–256 (1952). (Russian)

The paper discusses the "paradox" connected with the situation in which certain measurements carried out on one particle give information concerning a second particle [Einstein, Podolsky and Rosen, Physical Rev. (2) 47, 777–780 (1935); N. Bohr, ibid. 48, 696–702 (1935); other references in paper]. The author proposes a resolution of the paradox based on the standpoint that, if a wave function is a general function of the coordinates of two particles, then the two particles form a single system, i.e., there exists a bond between them even in the absence of ordinary forces. The remainder of the paper is devoted largely to criticisms of the discussion of the paradox by other writers.

*N. Rosen* (Haifa).

**Aleksandrov, A. D. On the meaning of wave function.** Doklady Akad. Nauk SSSR (N.S.) 85, 291–294 (1952). (Russian)

According to the author, there exist three points of view regarding the interpretation of the wave function of an electron in quantum theory: 1) The wave function characterizes the objective state of the electron; 2) it represents our knowledge of the state; 3) it refers not to an electron, but to an ensemble of electrons. In this paper the author supports the first point of view and opposes the other two.

*N. Rosen* (Haifa).

**Bhabha, H. J. Some new results on relativistic wave equations.** Report of an International Conference on Elementary Particles, Bombay, 1950, pp. 81–92; discussion, pp. 92–93. The International Union of Pure and Applied Physics, Bombay, 1952.

The author begins by reviewing his study [Rev. Modern Physics 17, 200–215 (1945); 21, 451–462 (1949); these Rev. 7, 272; 11, 764] of linear relativistic wave equations of the form $(\alpha^r p_r+\beta\kappa)\psi=0$. He discusses in some generality a particular case in which $\psi$ has 16 components belonging to a representation space of the complete Lorentz group, which decomposes into the direct sum $D(\frac{3}{2},\frac{1}{2})+D(\frac{1}{2},\frac{3}{2})$, where we use the notation of the related papers of Le Couteur [Proc. Roy. Soc. London. Ser. A. 202, 284–300, 394–407 (1950); these Rev. 12, 569]. He concludes that the degrees of the minimal equations of the $\alpha^r$ and of the generators $I^{rs}$ of the representation are not necessarily the same.

*A. J. Coleman* (Toronto, Ont.).

**Jehle, Herbert. Two-component wave equations.** Physical Rev. (2) 75, 1609 (1949).

The two component wave equation is written down and the transformation properties for all quantities given.

*L. Infeld* (Warsaw).

**Kilmister, C. W. Two-component wave equations.** Physical Rev. (2) 76, 568 (1949).

The author shows (in connection with Jehle's note) the relationship between the two component wave equation and Dirac's equation. *L. Infeld* (Warsaw).

**Serpe, J. Two-component wave equations.** Physical Rev. (2) 76, 1538 (1949).

It is shown that Jehle's equations describe Majorana particles interacting with a pseudo-vector field. *L. Infeld.*

**Gürsey, F. On two-component wave equations.** Physical Rev. (2) 77, 844–845 (1950).

Jehle's two component wave equations can be regarded as degenerate cases of the general wave equations.

*L. Infeld* (Warsaw).

**Raje, S. A. Linear meson wave equation in de Sitter space.** Progress Theoret. Physics 8, 384–385 (1952).

**Havas, Peter. The classical equations of motion of point particles. I.** Physical Rev. (2) 87, 309–318 (1952).

The classical equations of motion of particles interacting by means of neutral vector and scalar mesons are investigated. Using the field-theoretical method of Dirac [Proc. Roy. Soc. London. Ser. A. 167, 148–169 (1938)], equations of motion are found assuming the fields produced by particles to be purely retarded or half-retarded, half-advanced. Terms depending respectively on the previous or entire motion are found. Difficulties of interpretation lead to the suggestion that these terms be omitted. It is then shown that in the symmetric case the modified equations can be derived from a variation principle. Neither the original nor the modified retarded equations can be so derived.

*K. M. Case* (Ann Arbor, Mich.).

**Kamefuchi, Susumu, and Umezawa, Hiroomi. On the renormalization in quantum electrodynamics.** Progress Theoret. Physics 7, 399–405 (1952).

The authors present a new variant of the method of carrying out the renormalization program in quantum electrodynamics, differing only formally from earlier treatments. The idea is to avoid the use of explicit "counterterms" for cancelling the infinite self-energy and self-charge effects. They use the interaction representation in which the field operators obey the free-field equation for a particle of mass $m_0$, where $m_0$ is the mass of a "bare" electron and differs from the true electron mass $m$ by the self-energy $\delta m$. In this method the electron propagation-function is $S_F(m_0)$ instead of $S_F(m)$. By use of the identity

$$S_F(m_0)=S_F(m)/[1+2\pi i\delta m S_F(m)]$$

it can easily be shown that this treatment gives identically the same results as the usual method. *F. J. Dyson.*

**Källén, Gunnar. On the definition of the renormalization constants in quantum electrodynamics.** Helvetica Phys. Acta 25, 417–434 (1952).

A formulation of quantum electrodynamics in terms of the renormalized Heisenberg operators and the experimental mass and charge of the electron is given. The renormalization constants are implicitly defined and expressed as integrals over finite functions in momentum space. No discussion of the convergence of these integrals or of the existence of rigorous solutions is given. (From the author's summary.) *C. Kikuchi* (East Lansing, Mich.).

**de Broglie, Louis. Sur l'interprétation de la mécanique ondulatoire des systèmes de corpuscules dans l'espace de configuration par la théorie de la double solution.** C. R. Acad. Sci. Paris 235, 1345–1349 (1952).

**Vigier, Jean-Pierre. Mécanique ondulatoire dans l'espace de configuration.** C. R. Acad. Sci. Paris **235**, 1372–1375 (1952).

**Schrödinger, Erwin. Relativistic Fourier reciprocity and the elementary masses.** Proc. Roy. Irish Acad. Sect. A. **55**, 29–50 (1952).

It has been suggested that wave functions of particles satisfy equations of the type $F(\Box)\psi=\mu\psi$ where $\Box$ is the d'Alembertian operator and $F$ is a function of four variables which is its own Fourier transform or is an eigenfunction of a four-dimensional oscillator equation. The author studies functions in Minkowski space-time which satisfy the first of these properties and he also determines functions having the second property which is shown to be inequivalent to the first. As a result of these studies he states "that neither the demand of Fourier reciprocity nor the canonically reciprocal equation (5.1) [four-dimensional oscillator equation] appears to yield a sound mathematical basis for restricting the choice of the fundamental wave-operators of quantum mechanics in the desired way that would let us expect definite values for the ratios of the rest-masses of particles." *A. H. Taub* (Urbana, Ill.).

**Møller, C. Non-local field theory.** Report of an International Conference on Elementary Particles, Bombay, 1950, pp. 163–168; discussion, 169–172. The International Union of Pure and Applied Physics, Bombay, 1952.

It is shown that a non-local field theory which includes a scalar field and a spinor field in interaction is equivalent to a local field theory in which there is a non-localized interaction between different fields described by a form factor. *A. H. Taub* (Urbana, Ill.).

**Bechert, Karl. Ansätze zu einer nicht-linearen Elektrodynamik. II.** Ann. Physik (6) **10**, 430–448 (1952).

A theory of electrodynamics given in the author's earlier paper [Ann. Physik (6) **7**, 369–409 (1950); these Rev. **13**, 408] is extended. A quantization of the field is carried out by assuming

$$[p_k(r'),\ V_l(r)]=\frac{h}{i}\delta_{kl}\delta(r-r')$$

where $k, l=1, 2, 3$ and $p_k$ is the momentum conjugate to the velocity vector $V_l$. It is found that $V_4(r')$ does not commute with the space components of $p_k$. The invariance properties of the commutation rules are investigated. *C. Kikuchi.*

**Gião, Antonio. Quelques problèmes de physique théorique.** Gaz. Mat., Lisboa **12**, no. 50, 57–67 (1951).

In the present paper, several problems in theoretical physics are dealt with. These include the proper mass of the photon, the difference in the behavior of mass in a gravitational field and that of an electric charge in an electromagnetic field, properties of a generalized electromagnetic field, Schwinger-Tomonaga equation, and the periodic variations of stellar magnetic field. *C. Kikuchi.*

**Tanikawa, Yasutaka. Theory of super-quantization of quantized field and its applications.** Progress Theoret. Physics **7**, 193–206 (1952).

A method of "super-quantization", previously proposed by the author [same journal **5**, 692–695 (1950); these Rev. **12**, 465], is now refined. The method resembles Fock's second quantization [Z. Physik **75**, 622–647 (1932)]; "super-quantization" is applied to the state vectors of a quantized field. A symmetrical treatment of the four potentials in quantum electrodynamics without introduction of the indefinite metric for the scalar potential is thus made possible. Used in the regularized theory of a Bose field, the method leads to a physically consistent reinterpretation of the negative energy field. The vacuum of this field is defined as the state where only the zero-particle state is unoccupied while all the other states are occupied. When the zero-particle state is occupied, and one of the other states becomes vacant, a field with a positive energy particle distribution is obtained. Concerning the difficulties associated with the regularization of the Fermi field, the method offers no clue. *E. Gora* (Providence, R. I.).

**Galanin, A. D. Radiative corrections in quantum electrodynamics. I, II.** Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 448–461, 462–470 (1952). (Russian)

An investigation is presented in detail whose main results have been published earlier [Doklady Akad. Nauk SSSR (N.S.) **79**, 229–232 (1951); these Rev. **13**, 413]. In part I one-electron problems in absence of real photons are considered. In part II the method developed in I is applied to problems where an arbitrary number of electrons interact with real photons. *E. Gora* (Providence, R. I.).

***Heisenberg, W. Paradoxien des Zeitbegriffs in der Theorie der Elementarteilchen.** Festschrift zur Feier des zweihundertjährigen Bestehens der Akademie der Wissenschaften in Göttingen. I. Math.-Phys. Kl., pp. 50–64. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1951.

An expository article describing the author's point of view concerning the nature of causality and theories of elementary particles. For details see reviews of the author's earlier papers [Z. Naturforschung **5a**, 251–259, 367–373 (1950); **6a**, 281–284 (1951); these Rev. **12**, 573; **13**, 520]. *F. J. Dyson* (Ithaca, N. Y.).

**Jeffreys, Bertha. The classification of multipole radiation.** Proc. Cambridge Philos. Soc. **48**, 470–481 (1952).

A classification of electromagnetic radiation according to "multipole order" is obtained by using the exact expressions for the retarded potentials due to a vibrating charge-current distribution. The terms of these expansions belonging to an electric or magnetic $2^l$-pole are separated out and used to derive approximation formulas for large distances. Expressions for the rate of radiation are given in a form suitable for calculating quantum-mechanical transition probabilities. *E. Gora* (Providence, R. I.).

**Fubini, Sergio. Sull'equivalenza di due definizioni della matrice $S$.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 298–302 (1952).

Heisenberg's definition of the $S$-matrix is based on the time-independent Schroedinger equation, while Feynman and Dyson define the $S$-matrix as a unitary operator describing the state of the system at $t=\infty$ as a function of its state at $t=-\infty$. To show that the two methods are equivalent, the adiabatic hypothesis is used. *E. Gora.*

**Itô, Hirosi. On the density matrix in Hartree-field. I.** Progress Theoret. Physics **7**, 406–416 (1952).

A new proof is offered for Kofink's [Ann. Physik (5) **28**, 264–296 (1937)] form of the density matrix for a system of $n$ identical particles in a state belonging to a particular representation of the permutation group on $n$ objects.

Kofink's form has the advantage of giving the density explicitly in terms of the character of the representation.
*A. J. Coleman* (Toronto, Ont.).

**Kwal, Bernard. Les difficultés de la théorie du méson et des forces nucléaires.** Ann. Inst. H. Poincaré **12**, 207–222 (1951).

The ideas of a previous paper [J. Phys. Radium (8) **12**, 868–872 (1951); these Rev. **13**, 1012] are developed at length and explicit formulas are given for scalar, pseudoscalar, vector and pseudovector theories. *A. J. Coleman.*

**Wlassow, A. A. Der neue Inhalt des Mehrteilchenproblems.** Abh. Sowjet. Physik **2** = Sowjetwissenschaft, Beiheft **28**, 39–60 (1951).

Translated from Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **18**, 840–856 (1948); these Rev. **10**, 666.

**Jánossy, L. The passage of a wave packet through a potential barrier.** Acta Phys. Acad. Sci. Hungar. **2**, 171–174 (1952). (Russian summary)

It is shown that a one-dimensional wave packet colliding with an infinitely thin potential barrier splits as the result of the collision into two parts; analytic expressions for the fragments are given. *Author's summary.*

**Berger, J. M., Foldy, L. L., and Osborn, R. K. Equivalence theorems for pseudoscalar coupling.** Physical Rev. (2) **87**, 1061–1065 (1952).

The authors reverse the procedure so far used in establishing equivalence theorems for the coupling of pseudoscalar mesons and nucleons, and eliminate the pseudoscalar coupling term from the Hamiltonian in favor of a pseudovector coupling term by a suitable unitary transformation. The main argument in favor of this procedure is the appearance of a nonlinear coefficient in the pseudovector coupling term which may account for the saturation of nuclear forces and the independence of single nucleon motions in nuclei. This nonlinearity is linked up with a dependence of the effective rest mass of the nucleons on the meson potential $\varphi$: $M^*=(M^2+f^2\varphi^2)^{1/2}$ ($f$ coupling constant). The transformation leads to four additional terms: two of these are proportional to $(M^*-M)$, and vanish in neutral theories; one term represents a contact interaction; the remaining term combines with the Hamiltonian of the nucleons in such a way that a Hamiltonian is formed which corresponds to nucleons of effective rest mass $M^*$. These results show that the distinction between linear and nonlinear meson-nucleon coupling is largely arbitrary in that a linear coupling in one representation may be strongly nonlinear in another representation. The theory is worked out rigorously for two types of neutral fields, and for the charged and the symmetrical field. *E. Gora* (Providence, R. I.).

**Cap, Ferdinand. Über die statische Wechselwirkung von Leptonen vermittels eines de Broglie-Feldes.** Acta Physica Austriaca **6**, 35–44 (1952).

The interaction between leptons (electrons, positrons, and neutrinos) is ascribed to a de Broglie field (a Maxwell field with non-vanishing photon rest mass). One expects the Lagrangian of the interaction with such a field to have the form $-\epsilon s_\nu A_\nu + f n_{\mu\nu} f_{\mu\nu}$. Here $\epsilon$ and $f$ are coupling constants; $s_\nu$ and $n_{\mu\nu}$ are the four-vector and the six-vector which can be constructed from Dirac's spinor wave functions; the four-vector $A_\nu$ and the six-vector $f_{\mu\nu}$ represent the de Broglie field. The neutrinos interact with the other leptons only if $f\neq 0$; for practically vanishing photon mass, this interaction corresponds to axial dipole coupling. *E. Gora.*

**Nelipa, N. F. The quantum theory of a "radiating" electron.** Doklady Akad. Nauk SSSR (N.S.) **85**, 1259–1262 (1952). (Russian)

Covers the same point as two recent papers [Judd, Lepore, Ruderman, and Wolff, Physical Rev. (2) **86**, 123 (1952); Olsen and Wergeland, ibid. **86**, 123–124 (1952)]. The assertion of Parzen [ibid. **84**, 235–239 (1951)] that quantum mechanical corrections to the classical formula for the radiation of an accelerated point charge are sufficient to affect the design of synchrotrons, etc. is shown to be based on an incorrect approximation. The classical formula is shown to be well grounded theoretically, in agreement with the recent experiments of D. R. Corson [ibid. **86**, 1052–1053 (1952)].
*A. J. Coleman* (Toronto, Ont.).

**Širokov, Yu. M. A relativistic theory of free particles with three-dimensional extension.** Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 539–543 (1952). (Russian)

A counterexample is given to the widely held opinion that it is impossible to have a relativistic theory of a particle with three-dimensional extension. Let $u_i$ be the constant velocity vector ($u^iu_i=-1$) of a free particle, $\rho(y_i)$ an invariant, $y_i=x_i+u_iu^jx_j$, $T_{ik}=u_iu_k\rho(y_j)$. In the rest system of the particle, $y_s=x_s$ for $s=1, 2, 3$, $y_4=0$, and the energy momentum tensor reduces to the single component

$$T_{44}=-\rho(x_1, x_2, x_3, 0).$$

It is not apparent how this theory can be extended to interacting particles. *A. J. Coleman* (Toronto, Ont.).

**Širokov, Yu. M. On the spin of particles with zero rest-mass.** Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **23**, 78–82 (1952). (Russian)

As a continuation of his earlier work on spin [same Žurnal **21**, 748–760 (1951); these Rev. **13**, 610], the author investigates the spin of a particle having zero rest-mass. He finds that the spin operator is proportional to the energy-momentum operator, so that its square vanishes. The spin properties are determined by a pseudo-scalar operator which has no non-relativistic analog. It appears, according to the author, that for an irreducible representation this operator has only two eigenvalues, in agreement with the result obtained by Fierz [Helvetica Phys. Acta **13**, 45–60 (1940); these Rev. **1**, 352] that particles of zero mass can have only two non-equivalent states of polarization. *N. Rosen.*

**Brulin, O., and Hjalmars, S. Wave equations for integer spin particles in gravitational fields.** Ark. Fys. **5**, 163–174 (1952).

Corresponding to the extension of the Dirac equations for the electron to the general theory of relativity, as carried out by earlier writers [Schrödinger, S.-B. Preuss. Akad. Wiss. **1932**, 105–128; Bargmann, ibid. **1932**, 346–354], the authors carry out a similar extension of the Duffin-Kemmer equations [Duffin, Physical Rev. (2) **54**, 1114 (1938); Kemmer, Proc. Roy. Soc. London. Ser. A. **173**, 91–116 (1939); cf. also Petiau, C. R. Acad. Sci. Paris **200**, 374–375 (1935)] for particles of spin 0 and 1. *N. Rosen.* (Haifa).

**Rosen, Nathan. Interaction between electron and one-dimensional electromagnetic field.** Physical Rev. (2) **87**, 940–942 (1952).

The interaction of a single electron with a quantized transverse electromagnetic field is discussed. A transformed

Hamiltonian is used from which spatial electron coordinates have been eliminated. For simplicity only the "one-dimensional" solutions are considered: the interaction energy diverges. *C. Strachan* (Aberdeen).

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## *Thermodynamics, Statistical Mechanics*

**Mazur, P. Sur les états à production d'entropie minimum dans les systèmes continus.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 182–196 (1952).

The author's purpose is to generalize to the case of continuous media a theorem of Prigogine [Etude thermodynamique des phénomènes irréversibles, Desoer, Liège, 1947] to the effect that steady non-equilibrium states are characterized by a minimum of entropy production. He states that the paradoxical results of Haase [Z. Naturforschung **6a**, 522–540 (1951)] in this connection are due to an incorrect formulation of the principle. The author's method is to consider examples. First, in the case of simple heat conduction the rate of entropy production is proportional to the volume integral of $[\text{grad}\,(1/T)]^2$. Minimization leads to Fourier's equation for the steady case. The author shows also that when the temperature is constant on the boundary, in a non-steady state the rate of entropy production decreases steadily with the time. From this he concludes that the states of minimum entropy production are stable. The author's second example concerns electric currents; the third, thermal diffusion and chemical reactions.

*C. Truesdell* (Bloomington, Ind.).

**Egerváry, Jenő, and Turán, Pál. On some questions of the kinetic theory of gases.** Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 303–314 (1951). (Hungarian)

The authors continue their work started in an earlier paper [Studia Math. **12**, 170–180 (1951); these Rev. **13**, 761]. We use the notations explained in that review. Assume that the velocities of the $n$ particles satisfy for $t=0$

$$\dot{x}_{\nu,0}=n^{2/5}(1+\nu n^{-101/100}),\quad \dot{y}_{\nu,0}=n^{2/5}(1+\nu n^{-101/100})\sqrt{2},$$
$$\dot{z}_{\nu,0}=n^{2/5}(1+\nu n^{-101/100})\sqrt{3}.$$

Then the authors prove that for any $0\leqq t\leqq n^{1/4}$ except time intervals whose total length does not exceed $cn^{41/300}\log n$ we have for every $k$

$$\left|\frac{N(t_0,k)}{n}-\frac{v_k}{n^2}\right|\leqq n^{-1/10}.$$

*P. Erdös* (Los Angeles, Calif.).

**Kubo, Ryogo. An expansion theorem of the density matrix.** J. Chem. Phys. **20**, 770–777 (1952).

The author develops a formal expansion theorem of the density matrix of equilibrium quantum statistics, from a point of view different from that of recent work in this field [J. E. Mayer and W. Band, same J. **15**, 141–149 (1947); H. S. Green, ibid. **19**, 955–962 (1951); M. L. Goldberger and E. N. Adams, unpublished].

The author's theorem reads as follows, after removing certain obvious grammatical errors: Let the density matrix be

$$\rho\{M\}=\exp\left[-\beta\sum_{l=1}^{M}H_l\right]$$

where the [linear] operators $H_l$ are not always commutable. $\rho\{M\}$ can be expanded in a series of the form

$$\rho\{M\}=\sum_{M\geq\sum_{i=1}^{n}m_i,\,h\geq 0}\ \sum_{\{m_1\}_M\cdots\{m_n\}_M}\rho^*\{m_1\}\cdots\rho^*\{m_n\}\cdot\bar{\rho}\{M-m_1-\cdots-m_n\},$$

where $\{m_1\}_M,\cdots,\{m_n\}_M$ are subsets of operators chosen from the set $\{M\}$ and operators belonging to different subsets are commutable. The summation is over all possible choices of the sets $\{m_1\},\cdots,\{m_n\}$. Here $\bar{\rho}\{k\}$ is defined by

$$\bar{\rho}\{k\}=\Big[\prod_{\{k\}}\exp(-\beta H_j)\Big]_s,$$

where suffix $s$ means symmetrization by changing the order of the products, and $\rho^*\{m\}$ is defined by

$$\rho^*\{m\}=\sum_{\{k\}m}(-)^{m-k}\rho\{k\}\bar{\rho}\{m-k\}$$

which is proved to be $O(\beta^{m+1})$. He indicates several applications of this theorem to ferromagnetism and other topics.

*A. W. Sáenz* (Bloomington, Ind.).

**Nakamura, Tutô. Statistical mechanics of cooperative phenomena.** Progress Theoret. Physics **7**, 241–254 (1952).

The author shows how the formulation of cooperative phenomena due to Kirkwood [J. Chem. Phys. **3**, 300–313 (1935); **9**, 514–526 (1941); **10**, 394–402 (1942)], and set up originally for liquid phenomena, can be used as a basis for obtaining many of the methods used in crystal statistics such as that of Bethe, Fowler-Gugenheim and Kramers-Wannier. This Kirkwood method is based on integral equations and uses a parametric method of varying the number of atoms present. The basic expression for evaluating the partition function is that for an integral transform in which the kernel involves the energy of interaction between the atoms already present and the new ones added and a parameter which permits one to vary this interaction potential from zero to its final value. The author introduces a Taylor series expansion for this kernel in terms of this parameter and certain simplifications. The notation is not always explained and because of this the reviewer was not able to follow certain details of the discussion, although the development in general appears to be quite straightforward.

*F. J. Murray* (New York, N. Y.).